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THE FITTING IDEAL PROBLEM

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Dedicated to Wolmer Vasconcelos on his seventieth birthday

Abstract

Let A be a Noetherian local ring and let E be a finitely generated A-module having rank r. In this note we deal with the conjectured inequality $\ell(\bigwedge^r E) \geq \text{height}\,(\text{Fitt}_r(E))$, where $\text{height}\,\text{Fitt}_r(E)$ is the codimension of the rth Fitting ideal of E and $\ell(M)$ stands for the analytic spread of a module M. We establish both cases where the inequality holds and fails. A special case where the inequality holds implies the celebrated Zak inequality for the dimension of the image of the Gauss map.

1. *Introduction*

Let $(A, \mathfrak{m})$ stand for a Noetherian local ring and its maximal ideal and let E denote a finitely generated A-module having rank r. Two central numerical invariants of E are the codimension of the rth Fitting ideal $\text{Fitt}_r(E)$ and the analytic spread $\ell(\bigwedge^r E)$. While the first is classically understood as a measure of the size of the non-free locus of E, the second gives a somewhat subtler information regarding the maximal number of analytically independent elements of $\bigwedge^r E$ (see the next section). Establishing lower bounds for analytic spreads is an important task not only in commutative algebra, but also in algebraic geometry, where they often give the dimensions of fundamental varieties such as secant and tangential varieties, dual varieties and Gauss images, to name the most well-known ones.

The purpose of this note is to study the interplay of these two seemingly unrelated invariants. In this regard the following question was posed in [**10**, 3.4].

Problem 1. Let E be a finitely generated A-module having rank r such that $E/\tau(E)$ is not free. When does the inequality $\ell(\bigwedge^r E) \geq \text{height}\,(\text{Fitt}_r(E))$ hold?

Here $\tau(M)$ denotes the torsion submodule of an A-module M.

One reason for asking this question in [**10**] was that, in a particular setup, an affirmative answer implies the so-called Zak inequality

$$\dim \Gamma(X) \geq \dim X - \dim(\text{Sing}(X)) - 1, \tag{1}$$

where $X \subset \mathbb{P}^n = \mathbb{P}^n_k$ is a non-linear d-dimensional projective subvariety and $\Gamma(X) \subset \mathbb{G}(n,d)$ denotes the Gauss image of the embedding $X \subset \mathbb{P}^n$. For this one takes A and E as above to be the homogenous coordinate ring of the embedding $X \subset \mathbb{P}^n$ and the module $\Omega_{A/k}$ of Kähler k-differentials of A, respectively. In Subsection 3.1 we actually revisit the proof of the required inequality in this case by setting it up under a general principle that carries out some other results as well.

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An affirmative answer to Problem 1 would also imply an analogous inequality in the context of local Gauss maps. Namely, one now takes $(A, \mathfrak{m})$ to be a Noetherian local domain essentially of finite type over an algebraically closed field k and still considers $E = \Omega_{A/k}$ (only now notice that the rank of $\Omega_{A/k}$ is $\dim A + \operatorname{trdeg}_k(A/\mathfrak{m})$). In this context the problem is affirmatively answered when A is Cohen–Macaulay or quasi-Gorenstein (see [**10**, 3.3]), but otherwise entirely open.

It may be enlightening to point out that the method employed in [**10**] consisted of reducing the problem to the case where the rank one torsionfree module $\bigwedge^r E/\tau(\bigwedge^r E)$ is embedded as a particular ideal of height one. In fact, we will show in this work that, as a general guide, the problem can be reduced to the case of an ideal of height one and, moreover, inductively assume that this ideal is principal locally on the punctured spectrum $\operatorname{Spec}(A) \setminus \{\mathfrak{m}\}$. In this context, we prove both positive and negative cases of the main problem, in connection with the nature of the Picard group of the punctured spectrum of A.

Our philosophy is that, since the question has an affirmative answer for regular local rings – more generally, for unique factorization domains or even for parafactorial rings – the next case would be complete intersections and, luckily, Gorenstein local rings. Thus, we deal with such assumptions over rings of dimension up to 3 and obtain results where the question is affirmatively answered. In the complete intersection case we prove a general affirmative result in dimension ≥ 4.

These results are pretty tight in the sense that, if either the required hypotheses are missing or else the dimension of the ring is larger than 3, then the proposed inequality fails to take place. Our work shows that even among familiar rings there are examples of ideals $I \subset A$ for which the answer to Problem 1 is negative. These examples are optimal in the sense that they actually satisfy the inequality $\ell(I) < \min\{\dim A, \mu(I)\}$, where μ denotes minimal number of generators.

2. *Preliminaries and reduction principles*

We briefly recall the main tools around the notion of analytic spread. Our standing references are [**3**] and [**11**].

Let A be a Noetherian ring and let E be a finitely generated A-module having rank r – i.e., E_p is A_p-free of rank r for every prime $p \in \operatorname{Ass}(A)$ or, equivalently, $K \otimes_A E \simeq K^r$, where K denotes the total ring of quotients of A. In this context, we may define the *Rees algebra* $\mathcal{R}(E)$ of E to be the residue algebra of the symmetric algebra $\mathcal{S}(E)$ by its A-torsion submodule $\tau(\mathcal{S}(E))$ – the latter is a homogeneous ideal in the standard A-grading of $\mathcal{S}(E)$.

If A is moreover local with maximal ideal $\mathfrak{m}$, one can introduce the residue $A/\mathfrak{m}$-algebra $\mathcal{F}(E) = A/\mathfrak{m} \otimes_A \mathcal{R}(E)$, called the *special fiber ring* of $\mathcal{R}(E)$. The *analytic spread* of E is $\ell(E) := \dim \mathcal{F}(E)$ (Krull dimension). The analytic spread of a finitely generated A-module E having a rank defined this way is therefore an invariant of E and coincides with previous definitions ([**8**]). If the residue field $A/\mathfrak{m}$ is infinite, then $\ell(E)$ gives the smallest possible number of generators of a *reduction* of E, i.e., a submodule $U \subset E$ such that $E^{n+1} = UE^n$ in $\mathcal{R}(E)$ for $n >> 0$.

Note that, by definition, $\ell(E) = \ell(E/\tau(E))$. The requirement in Problem 1 that $E/\tau(E)$ is not free is essential. Indeed, if $E = F \oplus \tau(E)$, where F is free, it is readily shown that $\operatorname{Fitt}_r(E) = \operatorname{Fitt}_0(\tau(E))$. Therefore $\operatorname{height}(\operatorname{Fitt}_r(E)) = \operatorname{height}(0 :_A \tau(E))$ is fairly arbitrary while $\bigwedge^r E/\tau(\bigwedge^r E) = \bigwedge^r F$, hence $\ell(\bigwedge^r E) = \ell(\bigwedge^r F) = 1$. In fact, since

$$\ell(\bigwedge^r E) = \ell(\bigwedge^r (E/\tau(E))) \leq \mu(\bigwedge^r (E/\tau(E))),$$

one has to impose the stronger condition $\mu(\bigwedge^r(E/\tau(E))) \geq \text{height}\,(\text{Fitt}_r(E))$. As it will turn out though, sometimes the stronger assumption follows from the weaker requirement of $E/\tau(E)$ not being free.

Torsion killing reduction principle. If Problem 1 has an affirmative answer for $E/\tau(E)$ then it is affirmative for E.

Proof. First note that the assumption of the problem that $E/\tau(E)$ is not free remains unchanged. As for the analytic spreads, the surjective map $E \twoheadrightarrow E/\tau(E)$ induces a surjective map $\bigwedge^r E \twoheadrightarrow \bigwedge^r(E/\tau(E))$ of modules of the same rank. Therefore, $\mathcal{R}(\bigwedge^r(E)) \simeq \mathcal{R}(\bigwedge^r(E/\tau(E)))$, hence $\ell(\bigwedge^r E) = \ell(\bigwedge^r(E/\tau(E)))$.

Finally, since $E/\tau(E)$ is not free the ideal $\text{Fitt}_r(E/\tau(E))$ is a proper ideal. Clearly, the surjection $E \twoheadrightarrow E/\tau(E)$ implies an inclusion of the respective Fitting ideals, from which follows the inequality $\text{height}\,(\text{Fitt}_r(E/\tau(E))) \geq \text{height}\,(\text{Fitt}_r(E))$. □

For the second reduction principle we will need the following result.

LEMMA 2.1. *Let A be a Noetherian local ring, let E be a finitely generated A-module, and let $t \geq 1$ be an integer. If $\bigwedge^t E$ is free and nonzero (if $\bigwedge^t E/\tau(\bigwedge^t E)$ is free and nonzero) then E is free (then $E/\tau(E)$ is free, respectively).*

Proof. (According to Vasconcelos) We argue by induction on t. Since it is obvious for $t = 1$, we assume that $t \geq 2$. There is a surjective A-homomorphism $E/\tau(E) \otimes \bigwedge^{t-1} E/\tau(\bigwedge^{t-1} E) \twoheadrightarrow \bigwedge^t E/\tau(\bigwedge^t E)$. Since $\bigwedge^t E/\tau(\bigwedge^t E)$ is free and nonzero, we have a surjective A-homomorphism $E \otimes \bigwedge^{t-1} E \twoheadrightarrow A$. Composing this map with the map $E \to E \otimes \bigwedge^{t-1} E$ induced by tensoring with any element of $\bigwedge^{t-1} E$ gives an A-homomorphism $E \to A$. But since A is local, one of these maps has to be surjective. Thus, we have a splitting $E = A \oplus E'$. Applying $\bigwedge^t$ to this decomposition one sees that $\bigwedge^{t-1} E'$ is a direct summand of $\bigwedge^t E$. Therefore $\bigwedge^{t-1} E'$ is free ($\bigwedge^{t-1} E'/\tau(\bigwedge^{t-1} E')$ is free, respectively).

We now claim that $\bigwedge^{t-1} E'/\tau(\bigwedge^{t-1} E')$ is nonzero or, equivalently, $\bigwedge^{t-1} E' \otimes_A K \neq 0$, where K is the total ring of quotients of A. Since $E = A \oplus E'$, we obtain $\bigwedge^t E \otimes_A K \simeq (\bigwedge^{t-1} E' \otimes_A K) \oplus (\bigwedge^t E' \otimes_A K)$. However $\bigwedge^t E \otimes_A K \neq 0$ by our assumption, hence indeed $\bigwedge^{t-1} E' \otimes_A K \neq 0$. Now we can apply the induction hypothesis. □

Ideal reduction principle. If Problem 1 has an affirmative answer for $\bigwedge^r E/\tau(\bigwedge^r E)$ then it is affirmative for E.

Proof. Let E be a finitely generated A-module with rank r such that $E/\tau(E)$ is not free. Since taking Fitting ideals and forming wedges commute with localization, by Lemma 2.1 the two ideals $\text{Fitt}_r(E)$ and $\text{Fitt}_1(\bigwedge^r E)$ are equal up to radical, hence have the same height. On the other hand, by the same lemma, $\bigwedge^r E/\tau(\bigwedge^r E)$ is not free. Since we are assuming the statement of the problem valid for this module, we get $\ell(\bigwedge^r E/\tau(\bigwedge^r E)) \geq \text{height}(\text{Fitt}_1(\bigwedge^r E/\tau(\bigwedge^r E))) \geq \text{height}\,(\text{Fitt}_1(\bigwedge^r E)))$, the latter inequality following simply again from the surjection $\bigwedge^r E \twoheadrightarrow \bigwedge^r E/\tau(\bigwedge^r E)$. Therefore $\ell(\bigwedge^r E) \geq \text{height}\,(\text{Fitt}_r(E))$. □

Dimension reduction principle. If locally on the punctured spectrum of A, Problem 1 has an affirmative answer for the torsion free rank one module $\bigwedge^r E/\tau(\bigwedge^r E)$, one may suppose that $\bigwedge^r E/\tau(\bigwedge^r E)$ embeds as a non-principal height one ideal that contains a regular element and is principal locally on the punctured spectrum.

Proof. Write $I \subset A$ for some embedding of $\bigwedge^r E/\tau(\bigwedge^r E)$. Since I is not free one has $I \subsetneq A$, and as I has rank one it contains a regular element a. Multiplying I by a we may further assume that height $I = 1$.

Now let $P \in \operatorname{Supp}(A/\operatorname{Fitt}_1(I))$ be such that $P \neq \mathfrak{m}$. Then I_P is not A_P-free and, by the assumption on lower dimension,

$$\ell(I) \geq \ell(I_P) \geq \operatorname{height}(\operatorname{Fitt}_1(I_P)) = \operatorname{height}((\operatorname{Fitt}_1(I))_P) \geq \operatorname{height}(\operatorname{Fitt}_1(I)),$$

which would give an affirmative answer to the problem. Thus, one may assume that the support of $A/\operatorname{Fitt}_1(I))$ consists only of the maximal ideal $\mathfrak{m}$. □

Thus, if one is looking for an affirmative answer to Problem 1, one may and will assume that the A-module E is an ideal $I \subset A$ of height one, containing a regular element, and principal locally on the punctured spectrum. In this setup, an affirmative answer means that $\ell(I) = \dim A$. As $\ell(I) \leq \mu(I)$, the best one could hope for though is $\ell(I) = \min\{\dim A, \mu(I)\}$.

For convenience we restate the problem in this format.

Problem 2. Let A be a local ring and let $I \subset A$ be a non-principal ideal of height one, containing a regular element, and principal locally on the punctured spectrum. When is $\ell(I) = \dim A$?

3. *Affirmative cases*

3.1. *Using integral dependence of ideals*

In [**10**] the fine work was to show that the canonical class map is not integral unless the original (standard graded) ring is a polynomial ring. This was sufficient for the main inequality using a fundamental result to the effect that in an equidimensional and universally catenary local ring A, for any ideal I and for any associated prime P of $A/\bar{I}$ ($\bar{I}$ = integral closure of I), one has $\ell(I) \geq \operatorname{height} P$ (see, e.g., [**7**, 4.1] where a complete list of attributions is given).

The essentials of this method can be collected in the following abstract formulation:

Theorem 3.1. *Let A be an equidimensional and universally catenary local ring, let E be a finitely generated module having rank r, and let $\bigwedge^r E/\tau(\bigwedge^r E) \simeq I \subset A$ be any embedding. If I is not a reduction of the saturation $I : (\operatorname{Fitt}_1(I))^\infty$ then* Problem 1 *has an affirmative answer for E.*

Proof. By Lemma 2.1, E and I have the same free locus. In fact, by the ideal reduction principle, we may assume that $E = I$. We are to show that $\ell(I) \geq \operatorname{height}(\operatorname{Fitt}_1(I))$. Supposing to the contrary, we prove that J is integral over I, where $J = I : (\operatorname{Fitt}_1(I))^\infty$. For this it suffices to show integrality locally at any associated prime P of $A/\bar{I}$. By the result previously quoted, $\operatorname{height} P \leq \ell(I)$, and since $\ell(I) < \operatorname{height}(\operatorname{Fitt}_1(I))$, we have $\operatorname{height} P < \operatorname{height}(\operatorname{Fitt}_1(I))$. Therefore $J_P = I_P \subset \bar{I}_P$, as was to be shown. □

Remark 1. Note that the saturation $I : (\operatorname{Fitt}_1(I))^\infty$ is the largest ideal of A that coincides with I locally on the free locus of I. In the setting of Proposition 3.1 it can be replaced by either $I^{**} = \operatorname{Hom}(\operatorname{Hom}(I, A), A)$ or by $I^{\vee\vee} = \operatorname{Hom}(\operatorname{Hom}(I, \omega_A), \omega_A)$ when A has a canonical ideal ω_A, since either module embeds into $I : (\operatorname{Fitt}_1(I))^\infty$.

Next we give two affirmative cases based on this method.

Let A be a Noetherian ring and let E be a finitely generated module having a rank $r \geq 1$. Recall that $\det(E) = (\bigwedge^r E)^{**}$ and that E is said to be *orientable* if $\det(E) \simeq A$. The latter condition holds, for instance, when A is a unique factorization domain, or E has a finite free resolution, or E is an ideal module in the sense of [**11**, Section 5].

Corollary 3.2. *Let A be a local equidimensional and universally catenary ring and let E be a finitely generated module having rank r. If E is orientable then* Problem 1 *has an affirmative answer.*

Proof. By the ideal reduction principle we may assume that E is torsionfree of rank one and orientable, but not free. In particular, $E \neq E^{**}$. Then the image of the natural embedding $E \hookrightarrow E^{**} = A$ lies in the maximal ideal of A, hence is not a reduction of E^{**}. Now apply Proposition 3.1 (via Remark 1). □

Corollary 3.3. *Let A be a reduced $\mathbb{N}$-graded Noetherian ring over a field and let E be a finitely generated graded A-module having rank r that is generated by finitely many homogeneous elements of degrees $d_1 \leq \ldots \leq d_n$. If $\det(E)$ has initial degree strictly smaller than $d_1 + \ldots + d_r$, then* Problem 1 *has an affirmative answer for E.*

Proof. Say, $E = \sum_{i=1}^{n} Ax_i$ with $\deg(x_i) = d_i$. Then $\bigwedge^r(E) = \sum_{1 \leq j_1 < \ldots < j_r} Ax_{j_1} \bigwedge \ldots \bigwedge x_{j_r}$, where $\deg(x_{j_1} \bigwedge \ldots \bigwedge x_{j_r}) = d_{j_1} + \ldots + d_{j_r}$. On the other hand, one has the natural embedding into the double dual

$$\bigwedge^r E/\tau(\bigwedge^r E) \hookrightarrow (\bigwedge^r E/\tau(\bigwedge^r E))^{**} = (\bigwedge^r E)^{**} = \det(E).$$

But now the assumption on the initial degree of $\det(E)$ forbids the image of this embedding to be a reduction of $\det(E)$, because up to a shift, $\det(E)$ can be identified with a homogeneous ideal of the reduced ring A. The result then follows from Proposition 3.1 (via Remark 1). □

As a deep application of the method of this part, one has:

Theorem 3.4. *Let A be a standard graded domain over a field k and assume that k is algebraically closed in the quotient field K of A. If A is not a polynomial ring then the answer to* Problem 1 *is affirmative for $\Omega_{A/k}$, the module of Kähler differentials of A.*

Proof. For the details of the proof we refer to [**10**, 2.1]. There one uses the method of Corollary 3.3 to reduce the problem to showing that the initial degree of ω_A is strictly less than $d = \dim A$. □

3.2. *Affirmative cases over special rings*

Dimension at most 2

The first testing ground for Problem 1 is certainly that of two-dimensional local rings. One can easily prove the following result:

Proposition 3.5. *Let A be a Cohen–Macaulay local ring of dimension at most* 2*. Then* Problem 1 *has an affirmative answer.*

Proof. By the ideal reduction and the dimension reduction principles we may assume that E is isomorphic to a non-principal ideal $I \subset A$ of height one, locally principal on the punctured spectrum. In this situation we need to show that $\ell(I) = \dim A$. Since $1 = \operatorname{height} I \leq \ell(I) \leq \dim A \leq 2$, it suffices to consider the case where $\ell(I) = \operatorname{height} I = 1 < \dim A = 2$. Then I is equimultiple, and since I is certainly principal locally at its minimal primes of minimal height, it is generated by a regular sequence according to [**2**]. Therefore I is principal, a contradiction. □

The following example shows that Problem 2 may have an affirmative answer even if A is not Cohen–Macaulay.

Example 1. Let A stand for the localization of the homogeneous coordinate ring $S = k[x, y, z, w] = k[X, Y, Z, W]/P$ of a monomial curve in $\mathbb{P}^3$ at its irrelevant ideal. It is known or easy to see that S has at most two singular primes of height one, namely, (x, y, z) and (y, z, w). Suppose that at most one of these is singular, say, $I = (x, y, z)$ is the regular prime and denote still by I its image in A. Then I is a non-principal ideal of height one, locally principal on the punctured spectrum of A, having $\ell(I) = 2$.

The calculation is as follows. Since (x, y, z) is regular on A, the parameters of the curve can be taken to be of the form $(t^a, t^b u^{a-b}, tu^{a-1}, u^a)$. Then the defining ideal P of the curve contains the two forms $Z^b - YW^{b-1}, YZ^{a-b} - XW^{a-b}$ as one readily verifies. It clearly suffices to check that I is principal in the ring A_w which is immediate from the form of these two equations. To see the analytic spread, one argues that the special fiber of I is $k[T, U, V]/(U^{a-1} - T^{b-1}V^{a-b})$ as all the remaining generators of the presentation ideal of the Rees algebra of I belong to the maximal ideal of A.

Remark 2. If we do not require I to be at least principal locally at its associated primes then there are many examples of such ideals of height one in a two-dimensional Cohen–Macaulay local ring with $\ell(I) = 1$ and yet non-principal. These examples are fairly easy to write down over the local ring at the irrelevant ideal of the homogeneous coordinate ring of a curve as above.

Dimension 3

Theorem 3.6. *Let A be a Gorenstein local ring of dimension* 3 *and let $I \subset A$ be an ideal of height one and with $\mu(I) \geq 3$. Assume that I is not a principal ideal but is principal locally on the punctured spectrum, and that there exists an element $x \in I$ and a positive integer n such that*

(i) *x generates I at its minimal primes and is part of a minimal reduction of I*
(ii) *The unmixed component of $A/(x, I^n)$ is Cohen–Macaulay.*

Then Problem 2 *has an affirmative answer for I.*

Proof. Let J^{unm} denote the unmixed part of an ideal J and let " $\tilde{}$ " indicate residues modulo $(x) \colon I^n$. We may assume that the latter ideal is proper as otherwise $I^n \subset (x)$ would imply that I is principal by the choice of x as in (i). Now, since A is a Gorenstein ring and $A/(x, I^n)^{\mathrm{unm}}$ is Cohen–Macaulay (by (ii)), linkage theory shows that $\tilde{A} = A/(x) \colon I^n = A/(x) \colon (x, I^n)^{\mathrm{unm}}$ is again a Cohen–Macaulay ring (of dimension two). By (i), the ideal $I + ((x) \colon I^n)$ has height at least two, hence height $\tilde{I} \geq 1$ and $I \cap ((x) \colon I^n) = (x)$, thus yielding $\tilde{I} \simeq I/(x)$.

Now, suppose that $\ell(I) < \dim A = 3$. Then, again by (a), $\ell(\tilde{I}) \leq \ell(I) - 1 \leq 1$, hence in fact $\ell(I) = 2$ and $\tilde{I}$ is an equimultiple ideal of height one in $\tilde{A}$. Since $\tilde{I}$ is principal locally in codimension one and $\tilde{A}$ is is Cohen–Macaulay, again by [**2**] $\tilde{I} = I/(x)$ is principal. Then $\mu(I) \leq 2$, contradicting the assumption. □

Remark 3. Proposition 3.6 is best possible if $\dim A = 3$ even if A is a quadric hypersurface isolated singularity. This is a general principle that can be used whenever there is an element in the Picard group of the punctured spectrum of A having sufficiently small analytic spread (see Section 4).

Example 2. Let $A = k[x_1, x_2, x_3, x_4] = k[X_1, X_2, X_3, X_4]/(X_1X_4 - X_2X_3)$ and let $I = (x_1, x_2)^2$. To make it local, localize at the maximal ideal (x_1, x_2, x_3, x_4). Clearly, $\dim A = 3$ and $\mu(I) = 3$. Since (x_1, x_2) is a height one prime and A is an isolated singularity, it follows that (x_1, x_2), hence I as well, is principal locally on the punctured spectrum. Furthermore $\ell(I) = \dim k[x_1^2, x_1x_2, x_2^2] = 2$. Therefore, $\ell(I) < \min\{\dim A,\ \mu(I)\}$.

Of course, the problem is that there is no element $x \in I$ satisfying both conditions (i) and (ii) above. Note that $x = x_1^2$ is an element satisfying condition (i), but for no value of n is condition (ii) satisfied (note that for $n = 1$, $(x_1^2) : I = I$ and $A/I = k[X_1, X_2, X_3, X_4]/(X_1X_4 - X_2X_3, X_1^2, X_1X_2, X_2^2)$ is unmixed, but not Cohen–Macaulay).

Corollary 3.7. *Let A be a Gorenstein local ring of dimension 3 and let E be a finitely generated A-module having rank r such that $E/\tau(E)$ is not a free module. Assume that:*

(i) *$E/\tau(E)$ is free locally on the punctured spectrum and $\mu(\bigwedge^r E/\tau(\bigwedge^r E)) \geq 3$;*

(ii) *$\det(E)$ is a Cohen–Macaulay A-module.*

Then Problem 1 *has an affirmative answer for E.*

Proof. By the ideal reduction principle we may replace E by $\bigwedge^r E/\tau(\bigwedge^r E) \simeq I \subset A$, with I an ideal of height one. Now $\det(E) = I^{**} \simeq I^{\mathrm{unm}}$, the unmixed part of I. The assumption on $\det(E)$ then implies that A/I^{unm} is a Cohen–Macaulay ring. The assertion now follows from Proposition 3.6 with $n = 1$. □

Complete intersections of dimension at least 4

Recall that a local ring A is said to be a *complete intersection* if its completion is isomorphic to a regular local ring modulo an ideal generated by a regular sequence.

Proposition 3.8. *Let A be a local complete intersection of dimension at least* 4 *and let E be a finitely generated A-module having rank r such that $E/\tau(E)$ is not a free module, but is free locally on the punctured spectrum. Then* Problem 1 *has an affirmative answer for E.*

Proof. By the ideal reduction principle we may replace E by $\bigwedge^r E/\tau(\bigwedge^r E) \simeq I \subset A$, with I an ideal of height one. Notice then that I is not principal, but is principal locally on the punctured spectrum. By the Grothendieck–Lefschetz Theorem ([**4**]), A is parafactorial. Therefore $I = (x) \cap Q = x(Q : x)$, for some element $x \in A$ and an ideal Q primary to the maximal ideal of A. Since I is not principal, $x \notin Q$ and hence $Q : x$ is also primary for the maximal ideal. But then $\ell(I) = \ell(Q : x) = \dim A$. □

Remark 4. Note that as a direct consequence of Proposition 3.8, by the same token as Theorem 3.4, we recover again the Zak inequality (1) in the case of a smooth complete intersection of dimension ≥ 3 which is not a linear subspace.

4. *Negative examples in higher dimension*

For geometric purposes one takes A to be a standard graded domain over an algebraically closed field k. Suppose our ideal $I \subset A$ is a homogeneous ideal generated in fixed degree. Then such a minimal set of generators, say, in number of $m + 1$, defines a rational mapping $\mathbb{P}^n \supset X \dashrightarrow \mathbb{P}^m$, where $X = \mathrm{Proj}(A)$ and n is its embedding dimension.

In this language the ideal version of the Fitting ideal problem has the following formulation:

PROBLEM 3. Let $X \subset \mathbb{P}^n$ be an integral subvariety and let $H \subset X$ be a subscheme of codimension one, defining ideal-theoretically the base locus of a rational map $X \dashrightarrow \mathbb{P}^m$. Suppose that H is locally everywhere defined by a single equation. When does the image of this map have dimension $\min\{\dim X, m\}$?

To see the translation, one recalls that the image of the rational map as above is the subvariety whose homogeneous coordinate ring is $\mathcal{R}_A(I) \otimes_A k$, where $I \subset A$ is a saturated ideal defining H.

The failure of a positive answer in general to Problem 3 was communicated to us by F. Russo. We further developed his ideas to fit in the frame of generic semi-symmetric matrices, with a combinatorial flavor. As it turns, we show that if A is no longer a complete intersection (as in Proposition 3.8), then the analytic spread may not attain maximum value in *any* dimension ≥ 3 even for arithmetically Gorenstein isolated singularities. Russo's idea was to take for X the projection of the 2-Veronesean embedding $Y \subset \mathbb{P}^N$ of $\mathbb{P}^n$ from a point $p \in Y$ and for $H \subset X$, a suitable linear space of dimension $n-1$ (for example, the intersection of the tangent space to Y at p with the hyperplane target of the projection).

Here is the general algebraic picture.

PROPOSITION 4.1. Let k a field and let $n \geq 1$ be a fixed integer. Let B denote the determinantal ring $k[\mathbf{x}] = k[\mathbf{X}]/I_2(M_{n+1})$, where M_{n+1} denotes the $(n+1) \times (n+1)$ generic symmetric matrix in the variables $\mathbf{X}$ and $I_2(M_{n+1})$ denotes its ideal of 2-minors. Let $A \subset B$ denote the k-subalgebra generated by the residues of all variables except the last one. Then

(a) There is a k-algebra presentation

$$A \simeq k[\mathbf{X} \setminus \{X_{n(n+3)/2}\}]/I_2(N_{n+1}),$$

where N_{n+1} denotes the semi-symmetric matrix obtained from M_{n+1} by omitting its last row and $I_2(N_{n+1})$ denotes its ideal of 2-minors.

(b) A is a normal Cohen–Macaulay isolated singularity of dimension $n+1$, and it is Gorenstein if and only if $n = 1$ or $n = 3$.

(c) Let $I \subset A$ denote the ideal generated by the residues of the $\binom{n+1}{2}$ entries of M_{n+1} striking out the last row and column of M_{n+1}, as depicted below

$$\begin{array}{ccccc} \underline{X_0} & \underline{X_1} & \cdots & \underline{X_{n-1}} & X_n \\ & \underline{X_{n+1}} & \cdots & \underline{X_{2n-1}} & X_{2n} \\ & & \ddots & \vdots & \vdots \\ & & & \underline{X_{n(n+3)/2-2}} & X_{n(n+3)/2-1} \end{array}.$$

If $n \geq 2$ then I is a non-principal codimension one ideal, locally principal on the punctured spectrum of A such that A/I is a regular ring. Moreover, $\mu(I) = \binom{n+1}{2}$ and $\ell(I) = n < \min\{\dim A, \mu(I)\} = n+1$.

Proof. (a): Recall that the 2-minors of M_{n+1} form a Gröbner basis of the ideal they generate in any term order (cf., e.g., [**12**, 1]), in particular in the elimination order consisting of the degree lexicographic order having $X_{n(n+3)/2}$ as largest variable. Thus the 2-minors not involving $X_{n(n+3)/2}$ are a Gröbner basis of the elimination ideal $k[\mathbf{X} \setminus X_{n(n+3)/2}] \cap I_2(M_{n+1})$ and it follows that this ideal coincides with $I_2(N_{n+1})$.

(b) and (c): First note that the inclusion $A \subset B$ is birational since, for example, in the field of fractions of B one has $x_{n(n+3)/2} = x^2_{n(n+3)/2-1}/x_{n(n+3)/2-2}$. In particular, $\dim A = \dim B = n+1$.

Next recall that B is a Veronese subring of the polynomial ring over k in $n+1$ variables and notice that A is a monomial subring of B, hence can be regarded as a monomial subring of the same polynomial ring. But the proof of part (a) showed that the initial ideal of the toric ideal of A with respect to the lexicographic order is generated by squarefree monomials. By [**13**, 13.15], A is then normal, hence also Cohen–Macaulay according to [**5**].

To argue that A is Gorenstein if and only if $n = 1$ or $n = 3$, we may obviously assume that $n \geq 2$. The result is then given in [**1**, 2.4(b)] as applied to the case when the matrix has size $n \times (n+1)$ and $t = 2$ (notation as in *loc. cit.*).

Next note that we chose $I \subset A$ to be generated exactly by the (residues of the) variables - call them $\widetilde{\mathbf{X}}$ - of the initial submatrix M_n of M_{n+1}. These variables clearly contain the ideal $I_2(N_{n+1})$, hence A/I is a regular ring and, from part (b), the ideal I has codimension $\binom{n+1}{2} - (\binom{n+1}{2} - 1) = 1$.

Now we prove that I is principal locally on the punctured spectrum of A. To this end let $P \subset A$ be a prime ideal containing I, but not the homogeneous maximal ideal. We may assume that, say, $x_n \notin P$ and then some of the determinantal relations in the ring A readily show that $I_P = (x_0, \ldots, x_{n-1})_P$. Again using some of the relations one sees that $x_i \in x_0 A_P$ for $0 \leq i \leq n-1$. Thus I_P is indeed principal.

It will now easily follow that A is an isolated singularity. In fact, let P be a prime ideal of A not coinciding with the homogeneous maximal ideal. If $I \subset P$ then by the above I_P is generated by a single regular element and A_P/I_P is a regular ring. Thus A_P is regular as well. If $I \not\subset P$, the determinantal relations on the Veronese ring B show that $x_{n(n+3)/2} \in A_P$, hence $A_P = B_P$. Since $(\mathbf{X})B_P = B_P$ and B has an isolated singularity, the result follows in this case too.

It remains to argue for the value of the analytic spread. But, the latter is the dimension of the k-subalgebra

$$C := \frac{k[\widetilde{\mathbf{X}}]}{I_2(N_{n+1}) \bigcap k[\widetilde{\mathbf{X}}]} \subset \frac{k[\mathbf{X} \setminus \{X_{n(n+3)/2}\}]}{I_2(N_{n+1})},$$

and, we of course claim that it is just $k[\widetilde{\mathbf{X}}]/I_2(M_n)$ again! By the discussion in the proof of (a), we know that the 2-minors of N_{n+1} form a Gröbner basis of the ideal $I_2(N_{n+1})$. Therefore, the contraction of this ideal to the variables $\widetilde{\mathbf{X}}$ is generated by those 2-minors of N_{n+1} involving those sole variables. Thus indeed $\dim C = n$. □

REMARK 5. Taking $n = 3$ in the above proposition shows that Proposition 3.6 is no longer true in dimension 4.

REMARK 6. We note that because of the presentation in (a), the ring A falls within the class of determinantal rings of semi-symmetric matrices studied by Conca ([**1**]). In particular, it is a normal Cohen–Macaulay domain. Since A is but a special case in this class, we thought a direct argument to show these two properties was desirable. We note that the divisor class group of A is $\mathbb{Z}$ and the class of the above ideal I is the canonical class of A ([**1**, 2.4(b)]). This strongly corroborates the idea mentioned in Remark 3.

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