A NEW WAY TO ANALYZE PAIRED COMPARISON RULES

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Abstract. The importance of "paired comparisons," whether ordinal or cardinal, has led to the creation of several methodological approaches; missing, however, is a common way to compare the various techniques. To do so, an approach developed here creates a coordinate system for the "data space." The coordinates capture which data aspects satisfy a strong cardinal transitivity condition and which ones represent "noise;" i.e., this second component consists of highly cyclic aspects that are formed by the data. This coordinate system leads to a procedure to analyze paired comparison rules. The procedure is illustrated by analyzing a new paired comparison rule, by comparing behaviors of selective rules, and by answering concerns about the Analytic Hierarchy Process (AHP) (such as how to select an "appropriate" consistent data term for an inconsistent one). An elementary (and quick) way to obtain AHP weights is introduced.

1. Introduction

Ranking of alternatives, whether to obtain an ordinal ranking to indicate "which one is better" or a cardinal approach to display distinctions and priorities in terms of weights, is a need that cuts across all disciplines. But realities such as costs, complexities, and other pragmatic concerns may favor using paired comparison approaches to analyze differences among the alternatives. These are preferred procedures for aspects of statistics, psychology, engineering, economics, operations research, individual decisions, voting, and on and on.

With the many paired comparison methods, there is the need to determine which one is more appropriate based on, perhaps, the criterion that it generates more reliable outcomes. To analyze this question and to address mysteries such as the emergence of cycles, the methodology developed here moves the discussion a step above analyzing specific rules to create a tool to do so — a coordinate system for data spaces. The purpose of this system is to facilitate the analysis and comparisons of rules, to explain why different approaches can yield different answers, and to show how to modify rules to avoid certain problems.

This paper has three main themes. The primarily one (Section 4) is to create a coordinate system that holds for quite general choices of "data spaces." By this I mean that the structure holds for a variety of kinds of data; maybe it comes from subjective choices

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of comparing alternatives, a statistical study, experimental conclusions, engineering measurements, voting tallies, etc.

A problem with paired comparisons is that the data need not satisfy even a crude sense of transitivity. Yet, for the outcomes to be useful, whether expressed as ordinal rankings or cardinal values where the weights indicate levels of priority, the outcomes must satisfy some kind of transitivity. This reality suggests finding ways to separate those data components that support this aim from those components that can compromise the objective and even create paradoxical conclusions. An added advantage of this structure is that it provides a natural way to analyze different paired comparison rules.

A second objective (Section 2.2) is to discuss a particular way to make paired comparisons. While this approach is not new (e.g., a version designed for engineering decisions is described in Saari-Sieberg [8]), this is the first description of this approach that applies to a wider variety of data structures. This approach is then analyzed by using the procedure that follows naturally from the data coordinate system.

A third objective is to use this procedure to analyze paired comparison rules to examine the Analytic Hierarchy Process (AHP) (Section 2.1). While AHP (developed primarily by T. Saaty) appears to have delightful properties, there remain mysteries. The data coordinate system provides some answers.

2. Two different methods

To make the discussion concrete, two different rules are described. One (AHP) is based on the eigenvector structure of a matrix that is defined by the data. For reasons that will become clear, AHP can be thought of as a multiplicative rule. In contrast, the Borda assignment rule (BAR) introduced in Section 2.2 is additive.

2.1. **Analytic Hierarchy Process.**

The Analytic Hierarchy Process (AHP) ranks N alternatives based on how an individual assigns weights to each $\{A_i, A_j\}$ pair of alternatives. When comparing A_i with A_j, the assigned $a_{i,j}$ value is intended to indicate, in some manner, the multiple of how much A_i is preferred to A_j. Thus the natural choice for $a_{j,i}$ is the reciprocal

$$a_{j,i} = \frac{1}{a_{i,j}}. \tag{1}$$

To ensure consistency with Equation 1, let $a_{j,j} = 1$.

These terms define a $N \times N$ matrix $((a_{i,j}))$ of positive entries. According to the Perron-Frobenius Theorem (e.g., Horn and Johnson [4]), matrix $((a_{i,j}))$ has an unique eigenvector $\mathbf{w} = (w_1, \dots, w_N)$, $w_j > 0$, $j = 1, \dots, N$, associated with the matrix's sole positive eigenvalue. For each j, the normalized value of w_j (e.g., let $\sum_{j=1}^N w_j = 1$; I use $w_2 = 1$ as a numeraire) defines the weight, "intensity," or "priority" associated with alternative A_j.

A "consistency" setting for this $((a_{i,j}))$ matrix is defined to be where each triplet i, j, k, satisfies the expression

$$a_{i,j}a_{j,k} = a_{i,k}, \quad i, j, k = 1, \dots, N. \tag{2}$$

The power gained from Equation 2 is that (Saaty [9, 10])

$$a_{i,j} = \frac{w_i}{w_j}, \quad i, j = 1, \ldots, N. \tag{3}$$

Consistency, then, provides an interpretation for the $a_{i,j}$ values; they equal the natural quotients of the w_j weights. Multiplying the eigenvector with the matrix shows that should Equation 2 be satisfied, the eigenvalue equals N.

While the Equation 2 consistency condition creates the intuitive interpretation for the $a_{i,j}$ and w_j values as given by Equation 3, natural questions remain; some follow:

(1) Equation 2 simplifies computations; does this condition have other interpretations that would appeal to modeling concerns?

(2) The $a_{i,j}$ terms, $i < j$, define the vector $\mathbf{a} = (a_{1,2}, a_{1,3}, \ldots, a_{N-1,N}) \in \mathbb{R}_+^{\binom{N}{2}}$, which lists the entries in the upper triangular portion of $((a_{i,j}))$ according to rows. (Here $\mathbb{R}_+^{\binom{N}{2}}$ is the positive orthant of $\mathbb{R}^{\binom{N}{2}}$; i.e., $\mathbb{R}_+^{\binom{N}{2}} = \{\mathbf{a} \in \mathbb{R}^{\binom{N}{2}} \mid a_{i,j} > 0\}$. When using $\mathbf{a}$ in an expression requiring an $a_{i,j}$ term where $i > j$, replace $a_{i,j}$ with $\frac{1}{a_{j,i}}$.)

Equation 2 defines a $(N-1)$-dimensional submanifold of $\mathbb{R}_+^{\binom{N}{2}}$ that I denote by $\mathbb{SC}_N$ (for "Saaty consistency"). Is there a natural interpretation for $\mathbb{SC}_N$?

(3) As $\mathbb{SC}_N$ is lower dimensional, it is unlikely for the selected $a_{i,j}$ values to define a point $\mathbf{a} \in \mathbb{SC}_N$. What does such an $\mathbf{a}$ mean? Namely, as these $a_{i,j}$ terms fail to satisfy Equation 2, do they have a natural interpretation?

(4) There are many ways to define an associated $\mathbf{a}' \in \mathbb{SC}_N$ for $\mathbf{a} \notin \mathbb{SC}_N$. One could, for example, let $\mathbf{a}'$ be the $\mathbb{SC}_N$ point closest (with some metric) to $\mathbf{a}$. But the infinite number of choices of metrics permits a continuum of possible $\mathbf{a}'$ choices. Can a natural choice for $\mathbf{a}'$ be found and justified in terms of the AHP structure rather than with the arbitrary selection of a metric?

(5) Is there a simple way (other than computing eigenvectors) to find the w_j values, at least in certain settings?

These and other questions are answered in this paper.

2.2. **Borda assignment rule.** An alternative way to compare a pair of alternatives $\{A_i, A_j\}$ also assigns a numeric value to each alternative in a pair; e.g., a typical choice comes from some interval, say $[-m_1, m_2]$, which I normalize to $[0, 1]$. Let the intensity assigned to A_i over A_j be $b_{i,j} \in [0, 1]$ where $b_{i,j}$ represents the share of the $[0, 1]$ interval that is assigned to A_i; e.g., in a gambling setting, $b_{i,j}$ could represent the probability that A_i beats A_j. With this interpretation, the $b_{j,i}$ value assigned to A_j is

$$b_{j,i} = 1 - b_{i,j}. \tag{4}$$

As Equation 4 requires $b_{i,j} + b_{j,1} = 1$, it is natural to require $b_{j,j} = \frac{1}{2}$, $j = 1, \ldots, N$.

More general conclusions arise by emphasizing the differences between weights (rather than the actual weights). To convert to this setting, let

$$d_{i,j} = b_{i,j} - b_{j,i}, \text{ so } d_{i,j} = -d_{j,i}. \tag{5}$$

This expression requires $d_{j,j} = 0$, which leaves $\binom{N}{2}$ distinct $d_{i,j}$ values where $i < j$. These terms define $\mathbf{d} = (d_{1,2}, d_{1,3}, \dots, d_{N-1,N}) \in \mathbb{R}^{\binom{N}{2}}$. (No restrictions are imposed on the origin, sign, or magnitude of each $d_{i,j}$.) Thanks to Equation 5, the $d_{i,j} = -d_{j,i}$ terms are used interchangeably for convenience.

In what I call the *Borda assignment rule* (BAR), assign the value

$$b_i = \sum_{j=1, j\neq i}^{N} b_{i,j} \tag{6}$$

to A_i, $i = 1, \dots, N$. Because

$$b_{i,j} = (b_{i,j} - \frac{1}{2}) + \frac{1}{2} = \frac{1}{2}(b_{i,j} - b_{j,i}) + \frac{1}{2} = \frac{1}{2}d_{i,j} + \frac{1}{2} \tag{7}$$

and $b_{i,i} = \frac{1}{2}$, equivalent expressions for BAR are $b_i = \sum_{j=1}^{N}[b_{i,j} - \frac{1}{2}] + \frac{N-1}{2}$, $i = 1, \dots, N$, and the one used here,

$$b_i = \bar{b}_i + \frac{N-1}{2} \text{ where } \bar{b}_i = \frac{1}{2}\sum_{j=1}^{N} d_{i,j}, \quad i = 1, \dots, N. \tag{8}$$

The $\bar{b}_i$ term is used more often than b_i.

A special but important case of BAR (which provides its name) comes the part of voting theory that analyzes majority votes over pairs. Here, with m voters, $b_{i,j}$ is the fraction of all voters who prefer candidate A_i over A_j. The BAR values for this election (Equation 6) turn out to be equivalent to what is known as the "Borda Count" tallies (Saari [7]). As a $m = 100$ voter example, if the A_1:A_2 tally is 60:40, the A_1:A_3 tally is 45:55, and the A_2:A_3 tally is 70:30, then

$$\begin{array}{l} b_{1,2} = 0.60,\ b_{2,1} = 0.40,\ d_{1,2} = 0.20; \quad b_{1,3} = 0.45,\ b_{3,1} = 0.55,\ d_{1,3} = -0.10; \\ b_{2,3} = 0.70,\ b_{3,2} = 0.30,\ d_{2,3} = 0.40. \end{array} \tag{9}$$

In turn, this means that

(10)

$$\begin{array}{rll} b_1 = & b_{1,2} + b_{1,3} = 0.60 + 0.45 = 1.05 & \text{which equals } 1 + \bar{b}_1 = 1 + \frac{1}{2}(0.20 - 0.10) \\ b_2 = & b_{2,1} + b_{2,3} = 0.40 + 0.70 = 1.10 & \text{which equals } 1 + \bar{b}_2 = 1 + \frac{1}{2}(-0.20 + 0.40) \\ b_3 = & b_{3,1} + b_{3,2} = 0.55 + 0.30 = 0.85 & \text{which equals } 1 + \bar{b}_3 = 1 + \frac{1}{2}(0.10 - 0.40) \end{array}$$

While BAR is agnostic about the origin of the $b_{i,j}$ values, voting theory traditionally requires voters to have complete transitive preferences over the alternatives. This assumption leads to the actual definition of the Borda Count, which is to tally a voter's ballot by assigning $N - j$ points to the j^{th} positioned candidate; e.g., if $N = 4$, then 3, 2, 1, 0 points are assigned, respectively, to the top-, second-, third-, and bottom-ranked candidates. With $N = 3$, the tallying weights are 2, 1, 0. As these tallies are based on the number of voters, they must be scaled by the multiple of $\frac{1}{m}$ to obtain the Equation 6 value.

To illustrate, the Equation 9 values also arise with transitive voter preferences

(11)

Number	Ranking	Number	Ranking
30	$A_1 \succ A_2 \succ A_3$	15	$A_2 \succ A_1 \succ A_3$
30	$A_3 \succ A_1 \succ A_2$	25	$A_2 \succ A_3 \succ A_1$

Because A_1 is top-ranked 30 times and second-ranked 45 times, her Borda tally is

$$(2 \times 30) + (1 \times 45) = 105.$$

When scaled by $\frac{1}{m}$, or $\frac{1}{100}$, the resulting value equals that of b_1 in Equation 10. Similar computations hold for A_2 and A_3.

3. Creating connections; a data coordinate system

The promised coordinate system (to separate data into portions that satisfy a strong version of transitivity and portions that manifest cycles) will be used to discover properties of BAR (Section 5). Of interest, the same coordinate system will be used to extract and compare AHP properties with those of BAR (Section 5).

To do so, an isomorphic relationship is established between $\mathbb{R}_+^{\binom{N}{2}}$, the domain of AHP, and $\mathbb{R}^{\binom{N}{2}}$, the domain of BAR. With this relationship, AHP concerns can be transferred to the setting with the data coordinate system. If the conveyed problem can be solved within this coordinate system, the answer can be transferred back to the AHP setting. This is the manner in which answers are found for the questions raised earlier about AHP.

To define the isomorphism $F : \mathbb{R}_+^{\binom{N}{2}} \to \mathbb{R}^{\binom{N}{2}}$, let the $\mathbb{R}^{\binom{N}{2}}$ coordinates be given by $d_{i,j} \in \mathbb{R}$. The $F(\mathbf{a}) = \mathbf{d}$ definition is to let

$$d_{i,j} = \ln(a_{i,j}),\ 1 \leq i < j \leq N, \text{ so } F^{-1}(\mathbf{d}) = \mathbf{a} \text{ is given by } a_{i,j} = e^{d_{i,j}}. \tag{12}$$

Clearly, F is an isomorphism. Modifying the entries to handle those $a_{i,j}$ and $d_{i,j}$ terms with $i > j$ is done in the natural fashion; indeed, F^{-1} essentially converts the $d_{i,j} = -d_{j,i}$ condition (Equation 5) into a constraint that is equivalent to Equation 1.

For another transferred relationship, when Equation 2 is expressed in $F(\mathbf{a})$ terms as $\ln(a_{i,j}) + \ln(a_{j,k}) = \ln(a_{i,k})$, it is equivalent to

$$d_{i,j} + d_{j,k} = d_{i,k}, \quad \text{for all } i, j, k = 1, \dots, N. \tag{13}$$

Equation 13 plays a central role in what follows, so it is worth providing an interpretation in terms of "transitivity of rankings."

Strong transitivity: Transitivity requires that if $A_i \succ A_j$ and $A_j \succ A_k$, then $A_i \succ A_k$. Re-expressed in terms of $d_{i,j}$ values, the definition for transitivity becomes

$$\text{if } d_{i,j} > 0 \text{ and } d_{j,k} > 0, \text{ then it must be that } d_{i,k} > 0; \quad i, j, k = 1, \dots, N. \tag{14}$$

The Equation 13 cardinal expression is much stronger than Equation 14; it has the flavor of measurements along a line where the signed distance from point i to point j plus the signed distance from j to k equals the signed distance from i to k. In fact, Equation 13 restricts $\mathbf{d}$ to a $(N-1)$-dimensional linear subspace $\mathcal{ST}_N$ that I call the *strong transitivity plane:*

$$\mathcal{ST}_N = \{\mathbf{d} \in \mathbb{R}^{\binom{N}{2}} \mid d_{i,j} + d_{j,k} = d_{i,k} \text{ for all } i, j, k = 1, \dots, N\}. \tag{15}$$

This connection provides an interpretation for Saaty's Equation 2 consistency condition.

Theorem 1. *$F(\mathcal{SC}_N) = \mathcal{ST}_N$. Stated in words, $\mathcal{SC}_N$ and $\mathcal{ST}_N$ are isomorphic.*

Thus the Equation 2 consistency condition is equivalent to a "strong cardinal transitivity" constraint, which is an attractive modeling requirement.

4. Coordinate system for $\mathbb{R}^{\binom{N}{2}}$

Not only does F convert the nonlinear $\mathcal{SC}_N$ into the linear $\mathcal{ST}_N$, it reduces complexities by replacing multiplication with addition. The next step is to analyze how data belonging to different $\mathbb{R}^{\binom{N}{2}}$ subspaces affect paired comparison rules, such as BAR and AHP values.

By definition, vectors (i.e., data) in the $\mathcal{ST}_N$ plane in $\mathbb{R}^{\binom{N}{2}}$ satisfy a strong transitivity condition. This connection suggests developing a coordinate system for $\mathbb{R}^{\binom{N}{2}}$ in terms of a basis for $\mathcal{ST}_N$ and a basis for its orthogonal complement (denoted by $\mathcal{C}_N$). The approach developed here is motivated by results from voting theory (Saari [6, 7]).

The basis for $\mathcal{ST}_N$ comes from the following:

Definition 1. *For each $i = 1, \dots, N$, let vector $\mathbf{B}_i \in \mathbb{R}^{\binom{N}{2}}$ be where each $d_{i,j} = 1$ for $j \neq i$, $j = 1, \dots, N$, and each $d_{k,j} = 0$ if $k, j \neq i$.*

Before proving that these vectors define a basis for $\mathcal{ST}_N$ (Theorem 2), it is worth previewing the results with examples.

Example ($N = 3$): For $N = 3$, we have $\mathbf{d} = (d_{1,2}, d_{1,3}, d_{2,3})$, so $\mathbf{B}_1 = (1, 1, 0)$, $\mathbf{B}_2 = (-1, 0, 1)$, $\mathbf{B}_3 = (0, -1, -1)$. Because $\mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3 = \mathbf{0}$, the system is linearly dependent. Any two vectors are independent, so they span a two-dimensional space.

To prove that this two-dimensional space is the desired $\mathcal{ST}_3$, it must be shown that if $\mathbf{d}^* = \alpha\mathbf{B}_1 + \beta\mathbf{B}_2 = (\alpha - \beta, \alpha, \beta)$, then $d^*_{1,2} + d^*_{2,3} = d^*_{1,3}$; that is, $\mathbf{d}^*$ satisfies the defining condition for $\mathcal{ST}_3$ (Equation 13). The proof reduces to showing that $d^*_{1,2} + d^*_{2,3} = (\alpha - \beta) + (\beta)$ always equals $d^*_{1,3} = \alpha$, which is true.

The normal space to $\mathcal{ST}_3$ is spanned by $(1, -1, 1)$. While this vector does *not* satisfy Equation 13, it provides a convenient way to capture the differences between transitivity (Equation 14) and strong transitivity (Equation 13). Namely, if $\mathbf{d} \notin \mathcal{ST}_3$ satisfies Equation 14, the fact $\mathbf{d} \notin \mathcal{ST}_3$ requires $\mathbf{d}$ to fail the Equation 13 transitivity condition; $\mathbf{d}$ must have a component in the $(1, -1, 1)$ normal direction (which will be shown to be cyclic). Thus, normal transitivity is a combination of strong transitivity with some cyclic actions. A natural extension of this comment proves that an open region of ordinal transitive rankings in $\mathbb{R}^{\binom{3}{2}}$ surrounds the lower dimensional subspace $\mathcal{ST}_3$. (This holds for all N.)

Example ($N = 4$): For $N = 4$ where $\mathbf{d} = (d_{1,2}, d_{1,3}, d_{1,4}, d_{2,3}, d_{2,4}, d_{3,4})$, we have that

$$\mathbf{B}_1 = (1,1,1,0,0,0), \quad \mathbf{B}_2 = (-1,0,0,1,1,0), \quad \mathbf{B}_3 = (0,-1,0,-1,0,1), \quad \mathbf{B}_4 = (0,0,-1,0,-1,-1). \tag{16}$$

The sum of these four vectors equals zero, so at most three of them are linearly independent; any three are. Condition Equation 13 is easily verified in the same manner as above.

The next theorem asserts that these conditions hold for all values of $N \geq 3$.

Theorem 2. *For $N \geq 3$, any subset of $(N-1)$ vectors from $\{\mathbf{B}_i\}_{i=1}^N$ spans $\mathcal{ST}_N$.*

Proof. To prove that $\sum_{j=1}^N \mathbf{B}_j = \mathbf{0}$, notice that each $d_{i,j}$ coordinate in this summation has only two non-zero terms; one comes from $\mathbf{B}_i$ and the other from $\mathbf{B}_j$. Because $d_{i,j} = -d_{j,i}$, one term is $+1$ and the other is -1, which proves the assertion.

At most $(N-1)$ of the $\mathbf{B}_j$ vectors are linearly independent; by symmetry, it suffices to show that the first $(N-1)$ are. That is, if $\sum_{j=1}^{N-1} \alpha_j \mathbf{B}_j = \mathbf{0}$, it must be that $\alpha_j = 0$ for all choices of j. But if $\alpha_j \neq 0$, then $\alpha_j \mathbf{B}_j$ has a non-zero $d_{j,N}$ component. This $d_{j,N}$ component, however, is zero for all remaining $\mathbf{B}_i$ vectors, which proves the assertion.

It remains to show that $\mathbf{d}^*$ in the span of $\{\mathbf{B}_j\}_{j=1}^{N-1}$ satisfies the strong transitivity condition of Equation 13. To simplify the proof (by avoiding special cases for the index N), let

$$\mathbf{d}^* = \sum_{j=1}^N \beta_j \mathbf{B}_j.$$

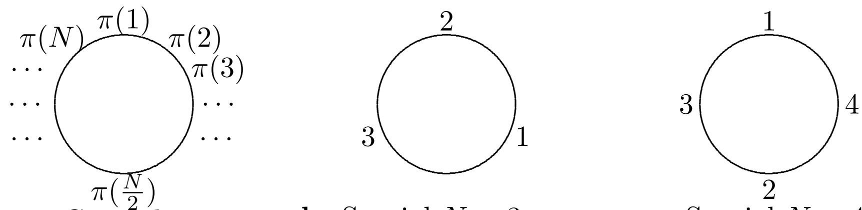
It must be shown that $d^*_{i,j} + d^*_{j,k} = d^*_{i,k}$ for all triplets $i, j, k = 1, \dots, N$. In this coordinate system, $d^*_{i,j} = [\beta_i - \beta_j]$ (To see this, if $i < j$, then $d^*_{i,j} = [\beta_i - \beta_j]$, so $d^*_{j,i} = -d^*_{i,j} = [\beta_j - \beta_i]$.) Thus $d^*_{i,j} + d^*_{j,k} = [\beta_i - \beta_j] + [\beta_j - \beta_k] = [\beta_i - \beta_k]$ which is the desired $d^*_{i,k} = [\beta_i - \beta_j]$ value.

As $\{\mathbf{B}_j\}_{j=1}^N$ spans a $(N-1)$-dimensional subspace of vectors that satisfy Equation 13, this set defines an $(N-1)$-dimensional subspace of $\mathcal{ST}_N$. But $\mathcal{ST}_N$ (defined by Equation 13) is a $(N-1)$-dimensional subspace, so $\{\mathbf{B}_j\}_{j=1}^N$ spans $\mathcal{ST}_N$. □

The $\mathcal{C}_N$ subspace. In determining the normal space for $\mathcal{ST}_N$, it will follow that these data portions always define cycles.

To define the basis (based on developments in [6, 7]) for this orthogonal, *cyclic space* $\mathcal{C}_N$, list the N indices, in any specified order, along the edges of a circle as indicated in Figure 1. In Figure 1a, $\pi(j)$ represents the integer listed in the j^{th} slot around the circle; Figures 1 b, c illustrate special cases. (In Figure 1b, $\pi(1) = 2, \pi(2) = 1, \pi(3) = 3$.) Moving in a clockwise motion about the circle, each integer is preceded by an integer and followed by a different integer.

Definition 2. *Let π be a specific listing, or permutation, of the indices $1, 2, \dots, N$ where the indices are listed in the cyclic fashion $(\pi(1), \pi(2), \dots, \pi(N))$ around a circle. Define $\mathbf{C}_\pi \in \mathbb{R}^{\binom{N}{2}}$ as follows: If j immediately follows i in a clockwise direction, then $d_{i,j} = 1$; if j immediately precedes i, then $d_{i,j} = -1$. Otherwise $d_{i,j} = 0$. Vector $\mathbf{C}_\pi$ is the "cyclic direction defined by π".*



a. General case **b.** Special $N = 3$ case **c.** Special $N = 4$ case

Figure 1. Cyclic arrangements of data

Example ($N = 3$)**:** Let $\pi = (2, 1, 3)$ as represented in Figure 1b. To compute $\mathbf{C}_\pi = (d_{1,2}, d_{1,3}, d_{2,3})$, because 2 precedes 1 on the circle, $d_{1,2} = -1$. Likewise, 3 immediately follows 1, so $d_{1,3} = 1$. For the remaining term of $d_{2,3}$, as 3 immediately precedes 2, we have that $d_{2,3} = -1$ Thus $\mathbf{C}_\pi = (-1, 1, -1)$.

Vector $\mathbf{C}_\pi$ significantly violates the strong transitivity of Equation 13. In sharp contrast to the $d_{1,2}+d_{2,3} = d_{1,3}$ requirement, the $\mathbf{C}_\pi$ entries are $d_{1,2} = d_{2,3} = -1$, where, rather than the required $d_{1,3} = -2$, we have $d_{1,3} = 1$. When expressed in terms of ordinal rankings, these entries represent the cycle $A_2 \succ A_1, A_1 \succ A_3$, and $A_3 \succ A_2$ each by the same $d_{i,j}$ difference. Indeeed, a direct computation proves that cyclic data direction $\mathbf{C}_\pi$ is orthogonal to both $\mathbf{B}_1 = (1, 1, 0)$ and $\mathbf{B}_2 = (-1, 0, 1)$, so $\mathbf{C}_\pi$ defines a normal direction for $\mathcal{ST}_3$.

By listing the indices around a circle, any rotation of these numbers does not affect which integers follow and precede a specified value; they all define the same $\mathbf{C}_\pi$ vector. Thus, each of the $(2, 1, 3)$, $(1, 3, 2)$ and $(3, 2, 1)$ orderings define the same $\mathbf{C}_{(2,1,3)} = (-1, 1, -1)$.

Three remaining orderings come from $(1, 2, 3)$ and its rotations. The cyclic direction representing these choices is the earlier $\mathbf{C}_{(1,2,3)} = (1, -1, 1) = -\mathbf{C}_{(2,1,3)}$. Thus, $\mathbf{C}_\pi$ spans the normal space $\mathcal{C}_3$ to $\mathcal{ST}_3$; it consists of data terms defining cyclic behavior.

Example ($N = 4$)**:** To underscore the simplicity of finding a $\mathbf{C}_\pi$, consider the permutation $\pi = (1, 4, 2, 3)$ (depicted in Figure 1c). Starting at the top, move in a clockwise direction: 4 follows 1 so $d_{1,4} = 1$; 2 follows 4 so $d_{4,2} = 1$ or $d_{2,4} = -1$; 3 follows 2 so $d_{2,3} = 1$, and 1 follows 3 so $d_{3,1} = 1$ or $d_{1,3} = -1$. All remaining $d_{j,k} = 0$, so the corresponding

$$\mathbf{C}_{(1,4,2,3)} = (d_{1,2}, d_{1,3}, d_{1,4}, d_{2,3}, d_{2,4}, d_{3,4}) = (0, -1, 1, 1, -1, 0). \tag{17}$$

The cyclic arrangement represented by $\mathbf{C}_\pi$ can also be read from Figure 1c; it is

$$A_1 \succ A_4, \quad A_4 \succ A_2, \quad A_2 \succ A_3, \quad A_3 \succ A_1$$

where each difference is captured by a common $d_{i,j}$ value.

The cyclic nature of the data represented by $\mathbf{C}_\pi$ suggests that $\mathbf{C}_\pi$ is orthogonal to $\mathcal{ST}_4$. To prove that it is, show that $\mathbf{C}_\pi$ is orthogonal to each $\mathbf{B}_j$ basis vector. The only non-zero entries of $\mathbf{B}_1$, for instance, are the $d_{1,j}$, $j > 1$, terms, which all equal unity. In the permutation π, "1" is immediately preceded and immediately followed by different integers; these $d_{1,j}$ terms will be, respectively, -1 and 1 in $\mathbf{C}_\pi$, thus they cancel in the scalar product with $\mathbf{B}_1$. As all other $d_{i,j}$ terms are zero in $\mathbf{C}_\pi$, orthogonality is verified.

Figure 1c represents one of $4! = 24$ permutations of the 4 indices; rotations of this ordering preserve which integers precede and follow others, so they all define the same

Equation 17 vector. Four other orderings are obtained by flipping the ordering (which is equivalent to using the ordering in a counter-clockwise manner). These orderings define a cyclic vector that is a (-1) multiple of the Equation 17 choice.

Three mutually orthogonal $\mathbf{C}_\pi$ vectors are orthogonal to $\mathcal{ST}_4$; they are given by

$$\mathbf{C}_{(1,2,3,4)} = (1, 0, -1, 1, 0, 1) = -\mathbf{C}_{(1,4,3,2)}, \quad \mathbf{C}_{(1,3,2,4)} = (0, 1, -1, -1, 1, 0) = -\mathbf{C}_{(1,4,2,3)},$$
$$\mathbf{C}_{(1,3,4,2)} = (-1, 1, 0, 0, -1, 1) = -\mathbf{C}_{(1,2,4,3)}$$

These vectors span the normal space – the cyclic space, $\mathcal{C}_4$. For a dimension count, $\mathbb{R}^{\binom{4}{2}}$ has dimension six and the (strongly transitive) subspace $\mathcal{ST}_4$ has dimension three, so the subspace $\mathcal{C}_4$ (spanned by the cyclic directions) accounts for the three remaining dimensions. This establishes a coordinate system for $\mathbb{R}^{\binom{4}{2}}$.

The next two statements extend this coordinate system to all choices of N.

Theorem 3. *For any $N \geq 3$, a $\mathbf{C}_\pi$ vector represents a particular cyclic ranking of the N alternatives $\{A_j\}$.*

For any triplet of distinct indices $\{i,\ j,\ k\}$, there exists a linear combination of the $\mathbf{C}_\pi$ vectors so that, rather than fulfilling Equation 13, the data satisfies

$$d_{i,j} = d_{j,k} = 1,\ d_{i,k} = -1,\ \textit{all other}\ d_{u,v} = 0. \tag{18}$$

Equation 18 corresponds to the cycle of the triplet

$$A_i \succ A_j, \quad A_j \succ A_k, \quad A_k \succ A_i,$$

each by the same $d_{i,j}$ difference; all other pairs are ties.

Proof. The $\mathbf{C}_\pi$ entries defined by permutation $(\pi(1), \pi(2), \dots, \pi(N))$ represents the cycle

$$A_{\pi(1)} \succ A_{\pi(2)}, \quad A_{\pi(2)} \succ A_{\pi(3)}, \quad \dots, \quad A_{\pi(N-1)} \succ A_{\pi(N)}, \quad A_{\pi(N)} \succ A_{\pi(1)}$$

each with the same $d_{i,j}$ difference. All remaining $d_{u,v}$ values equal zero, so all remaining pairs of alternatives are tied.

With the symmetry among indices, it suffices to prove Equation 18 by finding linear combinations of the cyclic directions so that $d_{1,2} = d_{2,3} = d_{3,1} > 0$ and all remaining $d_{i,j}$ values equal zero. For $N = 3$, the conclusion follows by using $\mathbf{C}_{(1,2,3)}$. For $N = 4$, the conclusion follows by using $\mathbf{C}_{(1,2,4,3)}$, $\mathbf{C}_{(1,2,3,4)}$, $\mathbf{C}_{(1,4,2,3)}$. In two of these arrangements, 1 is followed by 2, 2 by 3, and 3 by 1, so $d_{1,2} = d_{2,3} = d_{3,1} = 2$. For each of the three remaining pairs, $\{1, 4\}$, $\{2, 4\}$, $\{3, 4\}$, each index is immediately preceded in one permutation and immediately followed in another by the other index, so each $d_{u,v} = 0$.

For $N \geq 5$, consider the four permutations

$$\pi_1 = (1, 2, 3, r), \quad \pi_2 = (1, 2, 3, r'), \quad \pi_3 = (3, 1, s), \quad \pi_4 = (3, 1, s'),$$

where r, s are permutations of the remaining indices and the prime indicates the reversal of this listing. (So, if $r = (4, 6, 5)$, then $r' = (5, 6, 4)$.) The only restriction imposed on the r, s listings is that they must begin with 4 and end with 5.

To compute the $d_{i,j}$ values, each of $\{1, 2\}$, $\{2, 3\}$, and $\{3, 1\}$ appears, in this immediate order, in precisely two of the four listings; they never appear in a reversed order. Thus $d_{1,2} = d_{2,3} = d_{3,1} = 2$. To show that all remaining $d_{u,v} = 0$, the critical terms are $d_{4,j}$ and

$d_{5,j}$, $j = 1, 3$. (This is because these indices are adjacent in certain permutations.) To compute $d_{1,4}$, notice that 1 immediately precedes 4 in π_3 and immediately follows 4 in π_2; in all other listings the two indices are separated, so $d_{1,4} = 0$. A similar argument shows that $d_{1,5} = d_{3,4} = d_{3,5} = 0$. For any other pair of indices, if i and j are adjacent in r or s, then they are adjacent in the opposite order in the respective r', s', so $d_{i,j} = 0$. This completes the proof. □

Theorem 4. *For any $N \geq 3$ and permutation π of the indices, $\mathbf{C}_\pi$ is orthogonal to $\mathcal{ST}_N$. These $\mathbf{C}_\pi$ vectors span $\mathcal{C}_N$, which is the $[\binom{N}{2} - (N-1)] = \binom{N-1}{2}$-dimensional subspace of vectors normal to $\mathcal{ST}_N$.*

Proof. To prove that $\mathbf{C}_\pi$ is orthogonal to $\mathcal{ST}_N$, it suffices to show that $\mathbf{C}_\pi$ is orthogonal to each basis vector $\mathbf{B}_i$. The only non-zero entries in $\mathbf{B}_i$ are the $d_{i,j}$, $j \neq i$, terms, which all equal $+1$. But $\mathbf{C}_\pi$ has only two non-zero $d_{i,j}$ terms; one is for the j value immediately following i, and the other is for the j value immediately preceding i in the permutation. Because one entry is $+1$ and the other is -1, the scalar product is zero.

To show that the set $\{\mathbf{C}_\pi\}$ over all permutations of the indices π spans $\mathcal{C}_N$, it suffices to show that this set spans a linear subspace of dimension $\binom{N-1}{2}$. This is done by showing that the triplet cycles of Theorem 3 define $\binom{N-1}{2}$ linearly independent vectors. Key to the proof are the $d_{j,k}$ coordinates, $2 \leq j < k \leq N$; there are precisely $\binom{N-1}{2}$ of them. Define $\mathbf{V}_{j,k}$, $2 \leq j < k$, by the triplet $d_{1,j}, d_{j,k}, d_{1,k}$. Because index "1" occurs in two of these terms, $\mathbf{V}_{j,k}$ has unity in the $d_{j,k}$ component, and zeros for all other $d_{u,v}$, $u, v \geq 2$. Thus $\{\mathbf{V}_{j,k}\}_{2 \leq j < k \leq N}$ spans a $\binom{N-1}{2}$ dimensional subspace. This completes the proof. □

5. Examples, comparing decision rules

With its cyclic symmetry, no natural transitive (ordinal or cardinal) ranking can be assigned to $\mathcal{C}_N$ terms other than complete indifference. Combining this fact with the cyclic nature of the $\mathcal{C}_N$ data, it is reasonable to anticipate that these terms cause paired comparison paradoxes and different methods to have, with the same data, different answers. This is the case. In turn, a desired property of a paired comparison rule is if it can filter out all $\mathcal{C}_N$ content. As shown next, BAR does this.

5.1. BAR properties.

Basic properties of BAR follow almost immediately from the data coordinate system. To develop intuition about what to expect, consider the following general representation for $N = 3$:

$$\mathbf{d} = \beta_1\mathbf{B}_2 + \beta_2\mathbf{B}_2 + c\mathbf{C}_{(1,2,3)} = (\beta_1 - \beta_2 + c, \beta_1 - c, \beta_2 + c) \tag{19}$$

where $\mathbf{C}_{(1,2,3)} = (1, -1, 1) \in \mathcal{C}_3$. The BAR values are

$$\begin{aligned} \bar{b}_1 &= \tfrac{1}{2}[(\beta_1 - \beta_2 + c) + (\beta_1 - c)] = \tfrac{1}{2}[2\beta_1 - \beta_2], \\ \bar{b}_2 &= \tfrac{1}{2}[(\beta_2 - \beta_1 - c) + (\beta_2 + c)] = \tfrac{1}{2}[2\beta_2 - \beta_1], \\ \bar{b}_3 &= \tfrac{1}{2}[(-\beta_1 + c) + (-\beta_2 - c)] = -\tfrac{1}{2}(\beta_1 + \beta_2) \end{aligned} \tag{20}$$

Thus, according to Equation 20, $\bar{b}_j$ values are not affected, in any manner, by the $\mathcal{C}_3$ cyclic data components! As these terms cancel, BAR filters out this noise component.

Also, BAR values faithfully represent the strongly transitive $\mathcal{ST}_3$ components of the data. If, for example, $\beta_1 > \beta_2$, (so the strongly transitive *data* components reflect a preference for A_1 over A_2), then (Equation 20) BAR values rank $A_1 \succ A_2$ with the cardinal weights $\overline{b}_1 = \beta_1 - \frac{1}{2}\beta_2 > \overline{b}_2 = \beta_2 - \frac{1}{2}\beta_1$.[1] As shown in the next theorem, these BAR properties extend to all values of N.

Theorem 5. *For any $N \geq 3$, a general representation for $\mathbf{d} \in \mathcal{R}^{\binom{N}{2}}$ is*

$$\mathbf{d} = \sum_{j=1}^{N-1} \beta_j \mathbf{B}_j + \sum_{k=1}^{\binom{N-1}{2}} c_k \mathbf{C}_{\pi_k}, \tag{21}$$

where $\{\mathbf{C}_{\pi_k}\}_{k=1}^{\binom{N-1}{2}}$ spans $\mathcal{C}_N$.

For each $j = 1, \dots, N-1$,

$$2\overline{b}_j = (N-1)\beta_j - \sum_{k=1, k\neq j}^{N-1} \beta_k = \sum_{k=1, k\neq j} (\beta_j - \beta_k) \tag{22}$$

and $2\overline{b}_N = -\sum_{j=1}^{N-1} \beta_j$.

Basic BAR properties follow from Theorem 5. As catalogued next, for instance, $\overline{b}_j$ values strictly depend upon the data's strongly transitive components; the difference between any two $\overline{b}_j$ and $\overline{b}_i$ is not affected, in any manner, by $\mathcal{C}_N$ data cyclic components, nor even by β_k data values assigned to other alternatives. This heavy dependence of BAR on strongly transitive data components is reflected by the fact that $\overline{b}_i > \overline{b}_j$ and $A_i \succ A_j$ iff $\beta_i > \beta_j$.

Corollary 1. *For any $N \geq 3$, the $\overline{b}_j$ values are not influenced in any manner by the $\mathcal{C}_N$ cyclic data components; they are strictly determined by $\mathcal{ST}_N$ data components. All differences in the BAR values of alternatives faithfully reflect differences in the strongly transitive data components in that*

$$\overline{b}_j - \overline{b}_k = \frac{N}{2}(\beta_j - \beta_k). \tag{23}$$

Thus $A_j \succ A_k$ iff $\beta_j > \beta_k$.

As the above displays, BAR satisfies a list of strong, desired properties.

Proof. Equation 21 (Theorem 5) follows from the coordinate representation of $\mathbb{R}^{\binom{N}{2}}$.

The definition for $\overline{b}_j$ is linear (Equation 8), so it suffices to establish Equation 22 by computing $\overline{b}_j$ for each term in Equation 21. In computing the $\overline{b}_j$ value for a $\mathbf{C}_{\pi_k}$ term, notice that $\mathbf{C}_{\pi_k}$ has precisely two non-zero entries with the index j; one is accompanied with the index preceding j in the permutation, and the other is the index following j. Thus these two terms cancel in the computation of $\overline{b}_j$.

[1] Any basis other than $\{\mathbf{B}_1, \mathbf{B}_2\}$ has a similar representation. To see this, because $\mathbf{B}_3 = -(\mathbf{B}_1 + \mathbf{B}_2)$, if $\mathbf{d} = \beta_1\mathbf{B}_1 + \beta_2\mathbf{B}_2 = \tilde{\beta}_1\mathbf{B}_1 + \tilde{\beta}_3\mathbf{B}_3$, then $\tilde{\beta}_1 = \beta_1 - \beta_2$, $\tilde{\beta}_3 = -\beta_2$. Substituting into Equation 20 leads to $\overline{b}_1 = \frac{1}{2}[2\tilde{\beta}_1 - \tilde{\beta}_3]$, $\overline{b}_2 = -\frac{1}{2}(\tilde{\beta}_1 + \tilde{\beta}_3)$, and $\overline{b}_3 = \frac{1}{2}[2\tilde{\beta}_3 - \overline{\beta}_1]$; which is consistent with the Equation 20 form.

When computing $\bar{b}_j$ for $\beta_k\mathbf{B}_k$, the two cases are where $j = k$ and $j \neq k$. In the first setting, all $d_{j,i}$ terms equal unity, so these $N-1$ terms have the total contribution of $(N-1)\beta_j$. In the second case, $d_{j,k} = -1$ and all other terms involving j are zero; the contribution from this term is $-\beta_k$. (Recall, for $\mathbf{B}_k$, $d_{k,j} = 1$, so $d_{j,k} = -1$.) After including the $\frac{1}{2}$ multiple of Equation 8, Equation 22 follows. Equation 23 follows from Equation 21. □

5.2. **Other rules.** The following rules are selected to illustrate the $\mathbb{R}^{\binom{N}{2}}$ coordinate system. As it will become clear, a purpose of these rules (and many others) is to convert settings where the $d_{i,j}$ values define cycles into some form of transitive outcomes. These rules tend to become operative only with cyclic behavior and/or where the data fails to satisfy strong transitivity.

(1) The *Condorcet winner* (after Condorcet [2]) rule selects the alternative A_i for which $d_{i,j} > 0$ for all $j \neq i$. In other words, A_i beats all other alternatives.
(2) A method called here the *Kemeny approach* (as it resembles the method in Kemeny [5]) finds the "closest" transitive ranking to $\mathbf{d}$. This can be done by replacing certain $d_{i,j}$ values with $-d_{i,j}$ in a manner that satisfies a specified criterion; e.g., minimize the number of $d_{i,j}$ values that need to be reversed, minimize the sum of magnitudes of reversed numbers, etc.
(3) Whatever version of the Kemeny method is adopted, it can be extended to find a point $\mathbf{d}^* \in \mathcal{ST}_N$. $\mathbf{d}^*$ could be, for instance, the closest $\mathcal{ST}_N$ point to the data point obtained with the Kemeny method. Call this the *cardinal Kemeny method.*

Other rules can and have been developed, but the above suffice to demonstrate what happens. A way to demonstrate the role of $\mathcal{C}_N$ in affecting the outcomes of these rules is to show that even if $\mathbf{d}$ defines a transitive ranking, this ranking need not agree with the BAR values.

Example. With $N = 3$, let $\mathbf{d} = 3\mathbf{B}_1 + 2\mathbf{B}_2 = (1, 3, 2)$. Both $d_{1,2}$, $d_{1,3} > 0$, so $\mathbf{d}$ has A_1 ranked above A_2 and A_3. Now create

$$\mathbf{d}^* = \mathbf{d} + x(1, -1, 1) = (1 + x,\ 3 - x,\ 2 + x)$$

where $\mathbf{C}_{(1,2,3)} = (1, -1, 1) \in \mathcal{C}_3$ is a cyclic data component and the value of x is to be determined. For any x satisfying $-2 < x < -1$, the data has $d_{2,1}, d_{2,3} > 0$, so $\mathbf{d}^*$ has A_2, not A_1, ranked above both other alternatives. Thus all of the above rules favor A_2 over A_1 in their outcomes, But as shown above, BAR ignores the distorting cyclic terms, so it retains the original ranking with A_1 ranked above A_2. Cyclic $\mathcal{C}_N$ terms, in other words, are the components that can force different paired comparison rules to have different outcomes.

Extending the x values to $x < -2$ creates the cycle associated with $d_{1,2}, d_{2,,3} < 0$, and $d_{1,3} > 0$. In other words, the above change in the transitive ranking was only a preliminary stage in the continuum moving from the noise free (i.e., free of $\mathcal{C}_N$ terms) setting to cycles. (For $x > 3$, the opposite cycle (where $d_{1,2}, d_{2,3} > 0$, and $d_{1,3} < 0$) is created.)

As stated, the above rules kick in to handle settings where transitivity is violated and/or the weights do not satisfy strong transitivity. As one must anticipate that all reasonable rules agree for $\mathbf{d} \in \mathcal{ST}_N$, all differences arise with $\mathcal{C}_N$ components. A way to analyze a rule, then, is to compare its outcome for $\mathbf{d} \in \mathcal{ST}_N$ with its outcome for $\mathbf{d} + \mathbf{d}_1$ where $\mathbf{d}_1 \in \mathcal{C}_N$. The next result asserts that there can be differences, so the rule is affected by $\mathcal{C}_N$ terms. A following step (not done here) is to analyze these differences.

Theorem 6. *If $\mathbf{d} \in \mathcal{ST}_N$, the outcomes for the Condorcet, all versions of the Kemeny, and cardinal Kemeny methods agree with the BAR outcomes. Thus all differences these methods have in outcomes are completely due to components of $\mathbf{d} \notin \mathcal{ST}_N$ in the cyclic $\mathcal{C}_n$ direction. For each method, there exist $\mathbf{d}^*$ where its outcome disagrees with BAR.*

Proof. Clearly, all of these rules agree with $\mathbf{d} \in \mathcal{ST}_N$. It must be shown that there exist $\mathbf{d}^* \notin \mathcal{ST}_N$ where the rules disagree with BAR.

To construct an example, let $\mathbf{d}$ in Equation 21 have no cyclic terms and the β_j values satisfy $\beta_1 > \beta_2 > \cdots > \beta_{N-1}$, $\beta_2 > 0$, and $\beta_1 - \beta_2 < \beta_j - \beta_{j+1}$ for $j = 2, \ldots, N-2$. Next let $\mathbf{d}^* = \mathbf{d} + x\mathbf{C}_{(2,1,\ldots)}$ where x is to be determined; the only important part of the permutation is that 1 immediately follows 2. Thus, in the pairwise vote of $\mathbf{C}_{(2,1,\ldots)}$, the A_2:A_1 tally is $(N-1)$:1, with the difference $d_{2,1} = (N-2)$. By selecting x so that $\beta_1 - \beta_2 < x(N-2) < \beta_j - \beta_{j+1}, j = 2, \ldots, N-2$, it follows that $d_{1,2} > 0$ in $\mathbf{d}$ and $d_{1,2} < 0$ for $\mathbf{d}^*$. All remaining $d_{i,j}$ coordinates have the same sign in $\mathbf{d}$ and $\mathbf{d}^*$. In terms of ordinal rankings, $\mathbf{d}$ is accompanied by $A_1 \succ A_2 \succ A_3 \succ \cdots \succ A_N$, while $\mathbf{d}^*$ by $A_2 \succ A_1 \succ A_3 \succ \cdots \succ A_N$.

All of the specified methods, whether ordinal or cardinal, have the ranking of $\mathbf{d}^*$; the BAR outcome (because BAR cancels these cyclic terms) for $\mathbf{d}^*$ is that of $\mathbf{d}$. □

The problem, of course, is that even with the basic objective of these other methods to eliminate cyclic aspects from the outcome, problems arise because the "starting point" for these efforts remain strongly influenced by $\mathcal{C}_N$ terms.

5.3. **AHP outcomes; AHP and BAR connections.** AHP provides an excellent example to illustrate the procedure used to analyze rules. First, an appropriate coordinate system is developed to differentiate consistent from inconsistent forms of data. The system captures the type of noise distinguishing between the two setting, so it provides an interpretation for inconsistent data. Then, as above, AHP is analyzed by comparing what happens with consistent data $\mathbf{a}'$ and when $\mathbf{a}'$ is endowed with noise to create $\mathbf{a}$.

5.3.1. *When BAR and AHP agree.* To introduce the standard approach (of using F), consider the question whether AHP and BAR weights are related. Because F coverts $a_{i,j}$ values into $d_{i,j} = \ln(a_{i,j})$ terms, any such connection must relate $\bar{b}_j$ to $\ln(w_j)$ values. Indeed, the weight assigned by BAR to alternative A_i is determined by Equation 8, so

$$\bar{b}_i = \frac{1}{2}\sum_{j=1}^{N} d_{i,j} = \frac{1}{2}\sum_{j=1}^{N} \ln(a_{i,j}) = \frac{1}{2}\ln(\Pi_{j=1}^{N} a_{i,j}). \tag{24}$$

This expression extends part of Theorem 6 by showing that BAR and AHP outcomes agree at least in the favorable setting where Equation 2 is satisfied.

Theorem 7. *When the consistency Equation 2 is satisfied,*

$$\bar{b}_i = \frac{N}{2}\ln(w_i) - B, \quad i = 1, \dots, N, \tag{25}$$

where B is a constant.

Proof. If Equation 2 is satisfied, then $a_{i,j} = \frac{w_i}{w_j}$. It follows from Equation 24 that

$$\bar{b}_i = \frac{N}{2}\ln(w_i) - B, \quad B = \frac{1}{2}\ln(w_1 w_2 \dots w_N) \text{ for } i = 1, \dots, N,$$

which is Equation 25. Thus the w_i weights are common positive multiples of

$$w_i = exp(\bar{b}_i)^{\frac{2}{N}}, \quad i = 1, \dots, N. \tag{26}$$

Equation 26 will used to describe AHP settings for inconsistent data. □

By relating the AHP and BAR weights, the fairly extensive literature (e.g., see [7] and its references) establishing advantages of the Borda approach become available to AHP. In voting theory, for example, the Borda Count is the unique positional method (tally ballots by assigning specified weights to candidates based on their position on a ballot) that minimizes the numbers and kinds of consistency paradoxical outcomes that can occur. Many of these positive properties transfer, via the isomorphism, to provide new types of support, or maybe criticism, for AHP.

5.3.2. *Nonlinear coordinate system.* To preview the AHP nonlinear coordinate system, the $N = 3$ nonlinear coordinate system is developed first. Express Equation 21 as

$$\mathbf{d} = \sum_{j=1}^{2} \ln(\beta_j)\mathbf{B}_j + \ln(c)\mathbf{C}_{(1,2,3)}, \tag{27}$$

where $\beta_1, \beta_2, c > 0$ and $\mathbf{C}_{(1,2,3)} = (1, -1, 1)$. The $d_{1,2}$ term is $\ln(\beta_1) - \ln(\beta_2) + \ln(c) = \ln(\frac{c\beta_1}{\beta_2})$. In this manner, the general form for $\mathbf{d}$ becomes:

$$\mathbf{d} = (\ln(\frac{c\beta_1}{\beta_2}), \ln(\frac{\beta_1}{c}), \ln(c\beta_2)),$$

which means that the representation for $\mathbf{a} \in \mathbb{R}_+^{\binom{3}{2}}$ is

$$\mathbf{a} = (\frac{c\beta 1}{\beta_2}, \frac{\beta 1}{c}, c\beta_2), \quad \beta_1, \beta_2, c \in (0, \infty). \tag{28}$$

Each $\mathbb{R}^3$ point has a unique Equation 27 representation, so (from the properties of F) each $\mathbb{R}^3_+$ point has a unique Equation 28 representation.

Equation 28 reflects AHP features by describing a natural progression from where Equation 2 is satisfied ($c = 1$) to all levels of inconsistencies defined by various $c \neq 1$ values. A leaf of this foliation of $\mathbb{R}^3_+$ describes all $\mathbf{a}$ values that are associated with a specified c level; a leaf is given by

$$\mathcal{L}_c = \{ (\frac{c\beta 1}{\beta_2}, \frac{\beta_1}{c}, c\beta_2) \,|\, \beta_1, \beta_2 \in (0, \infty) \text{ for a fixed } c\}. \tag{29}$$

These leafs nicely mimic the structure of $\mathcal{SC}_3$ (where $c = 1$).

Re-expressing Equation 28 in terms of an $\mathbf{a}' = (a'_{1,2}, a'_{1,3}, a'_{2,3}) \in \mathcal{SC}_3$ (thus there are unique β_j values so that $a'_{1,2} = \frac{\beta_1}{\beta_2}$, $a'_{1,3} = \beta_1$, $a'_{2,3} = \beta_2$) creates the following statement.

Theorem 8. *For $\mathbf{a} \in \mathbb{R}_+^{\binom{3}{2}}$, there is a unique $\mathbf{a}' \in \mathcal{SC}_3$ and c defining*

$$\mathbf{a} = (a'_{1,2}c,\ \frac{a'_{1,3}}{c},\ a'_{2,3}c). \tag{30}$$

In words, an $\mathbf{a} \notin \mathcal{SC}_3$ can be interpreted as a distorted version of a consistent $\mathbf{a}' \in \mathcal{SC}_3$; the distortion is caused by the cyclic noise component identified by c. The noise, the c terms, twists an $\mathbf{a}' \in \mathcal{SC}_3$ to generate associated $\mathbf{a}$ terms throughout $\mathbb{R}_+^{\binom{3}{2}}$. The consistent $\mathbf{a}' = (6, 2, \frac{1}{3})$, for instance, defines the curve $\{(6t, \frac{2}{t}, \frac{t}{3}) \,|\, 0 < t < \infty\}$; all points on this curve of infinite length are distorted versions of $\mathbf{a}'$.

Similar comments and representations hold for $N \geq 3$. A word of caution; each $\mathcal{C}_N$ basis defines a different representation for the $\mathbb{R}_+^{\binom{N}{2}}$ nonlinear structure. All choices are equivalent, which permits selecting a choice that is convenient for a particular problem. Indeed, the Theorem 9 selection of triplets (described in Theorem 3 and used in the proof of Theorem 4) was chosen primarily to simplify the "bookkeeping" of the indices.

Theorem 9. *For $N \geq 3$, a nonlinear coordinate system of $\mathbb{R}_+^{\binom{N}{2}}$ that reflects the structure of AHP is given by the $N-1$ positive parameters $\beta_1, \dots, \beta_{N-1}$ and $\binom{N-1}{2}$ positive $\{c_{i,j}\}_{2 \leq i < j \leq N}$. They are combined to create the coordinates in the following manner:*

- *For $2 \leq i < j < N$, let $a_{i,j} = \frac{\beta_i}{\beta_j} c_{i,j}$.*
- *For $2 \leq i < N$, let $a_{i,N} = \beta_i c_{i,N}$.*
- *For $1 < j \leq N$ let $a_{1,j} = \frac{\beta_1}{\beta_j} C_{1,j}$ where $C_{1,j} = \frac{\Pi_{k>j} c_{j,k}}{\Pi_{k<j} c_{k,j}}$.*

These coordinates satisfy the AHP consistency equation, Equation 2, iff all $c_{j,k} = 1$.

Theorem 9 leads immediately to the following Corollary 2. Corollary 2 asserts that each $\mathbf{a} \in \mathbb{R}_+^{\binom{N}{2}}$ is constructed with a unique $\mathbf{a}' \in \mathcal{SC}_N$ and (after selecting the form of the noise; the triplet form is used here) a unique choice and structure for the associated noise. The contributions of this corollary are similar to those for $N = 3$; it provides a natural interpretation for an inconsistent $\mathbf{a}$ (in terms of the cyclic noise distorting the associated base $\mathbf{a}'$ values), it provides a foliation of the full space that can be used to understand the levels and kinds of inconsistencies (generated by different $c_{i,j}$ combinations and values), and so forth. As the results are similar, they are not repeated.

Corollary 2. *For $\mathbf{a} \in \mathbb{R}_+^{\binom{N}{2}}$, there exists a unique $\mathbf{a}' \in \mathcal{SC}_N$ and $\binom{N-1}{2}$ values of $c_{i,j} > 0$ so that for $2 \leq i < j < N$, $a_{i,j} = a'_{i,j} c_{i,j}$; for $2 \leq i < N$, $a_{i,N} = a'_{i,N} c_{i,N}$, and for $i = 1 < j \leq N$ $a_{1,j} = a'_{i,j} C_{1,j}$ where $C_{1,j} = \frac{\Pi_{k>j} c_{j,k}}{\Pi_{k<j} c_{k,j}}$.*

Proof. The $\mathbb{R}_+^{\binom{N}{2}}$ coordinate representation follows directly from the $\mathbb{R}^{\binom{N}{2}}$ coordinate representation

$$\mathbf{d} = \sum_{i=1}^{N-1} \ln(\beta_i)\mathbf{B}_i + \sum_{1<i<j} \ln(c_{i,j})\mathbf{C}_{i,j} \tag{31}$$

where scalars b_i, $c_{i,j} > 0$, and the vectors $\mathbf{C}_{i,j}$ represent the cyclic terms given by triplets of the $\{1, i, j\}$ form. Vector $\mathbf{C}_{i,j}$ has only three non-zero terms; two are where the $\{1, i\}$ and $\{i, j\}$ terms equal unity while the third, the $\{1, j\}$ term, equals -1. According to the structure of these basis vectors, the $1 < i < j < N$ term becomes $d_{i,j} = \ln(\frac{\beta_i c_{i,j}}{\beta_j})$, which leads to the $a_{i,j}$ representation. The only difference for $1 < i < N$ is that there are no β_N values, so $d_{1,N} = \ln(\beta_i c_{i,N}))$.

Some bookkeeping of indices is needed to handle the $1 < j$ setting. For each k, $1 < k < j$, the index pairs $\{1, k\}$ and $\{k, j\}$ have a positive entry in $\mathbf{C}_{k,j}$, while $\{1, j\}$ has the negative value. Thus the cyclic term adds $-\ln(c_{k,j})$ to the $\ln(\frac{\beta_1}{\beta_j})$ value. In the other direction, for each k, $1 < j < k$, the $\{1, j\}$ index pair defines a positive entry, so the $\ln(\frac{\beta_1}{\beta_j})$ value is changed by adding the $\ln(c_{j,k})$ values. Combining these logarithmic terms and transferring them via F leads to the Theorem 9 representation.

Each point in $\mathbb{R}_+^{\binom{N}{2}}$ is uniquely represented by choices of $\beta_j, c_{i,j}$; this is because Equation 31 is a coordinate system for $\mathbb{R}^{\binom{N}{2}}$ and the two spaces are isomorphic.

It is immediate to prove that these coordinates satisfy Equation 2 iff all $c_{j,k} = 1$. □

Example ($N = 4$): A nonlinear coordinate system for $N = 4$ is

$$\mathbf{a} = (a'_{1,2}xy, \quad a'_{1,3}\frac{z}{x}, \quad \frac{a'_{1,4}}{yz}, \quad a'_{2,3}x, \quad a'_{2,4}y, \quad a'_{3,4}z), \tag{32}$$

where $x = c_{2,3}$, $y = c_{2,4}$, $z = c_{3,4}$ and where

$$\mathbf{a}' = (a'_{1,2},\, a'_{1,3},\, a'_{1,4}, a'_{2,3},\, a'_{2,3},\, a'_{3,4}) = (\frac{\beta_1}{\beta_2},\, \frac{\beta_1}{\beta_3},\, \beta_1,\, \frac{\beta_2}{\beta_3},\, \beta_2,\, \beta_3) \in \mathcal{SC}_4$$

satisfies Equation 2; all $\beta_j, c_{j,k}$ values are positive.

As an illustration, the $\beta_1 = 2$, $\beta_2 = 3$, $\beta_3 = 4$ values define $\mathbf{a}' = (\frac{2}{3}, \frac{1}{2}, 2, \frac{3}{4}, 3, 4) \in \mathcal{SC}_4$. Thus the three-dimensional surface defined by the expression

$$\mathbf{a}_{x,y,z} = (\frac{2}{3}xy,\, \frac{1}{2}\frac{z}{x},\, \frac{2}{yz},\, \frac{3}{4}x,\, 3y,\, 4z), \quad 0 < x, y, z < \infty$$

represents all possible "noise" distortions of the original $\mathbf{a}'$. Without the coordinate system (Equation 32), there is no way (that I can see) to realize that $\mathbf{a}_{2,3,3} = (4, \frac{3}{4}, \frac{2}{9}, \frac{3}{2}, 9, 12)$ is AHP related to $\mathbf{a}_{3,2,1} = (4, \frac{1}{6}, 1, \frac{9}{4}, 6, 4)$ in that both share the same base $\mathbf{a}' = (\frac{2}{3}, \frac{1}{2}, 2, \frac{3}{4}, 3, 4)$.

Changing the $\mathcal{C}_4$ basis changes the representation. Using, $t\mathbf{C}_{(1,2,3,4)} + u\mathbf{C}_{(1,3,2,4)} + v\mathbf{C}_{(1,3,4,2)}$, for example, replaces Equation 32 with

$$\mathbf{a} = (a'_{1,2}\frac{t}{v}, \quad a'_{1,3}uv, \quad a'_{1,4}\frac{1}{tu}, \quad a'_{2,3}\frac{t}{u}, \quad a'_{2,4}\frac{u}{v}, \quad a'_{3,4}tv) \tag{33}$$

and the matrix

$$B = \begin{pmatrix} 1 & \frac{w_1}{w_2}\frac{t}{v} & \frac{w_1}{w_3}uv & \frac{w_1}{w_4}\frac{1}{tu} \\ \frac{w_2}{w_1}\frac{v}{t} & 1 & \frac{w_2}{w_3}\frac{t}{u} & \frac{w_2}{w_4}\frac{u}{v} \\ \frac{w_3}{w_1}\frac{1}{uv} & \frac{w_3}{w_2}\frac{u}{t} & 1 & \frac{w_3}{w_4}tv \\ \frac{w_4}{w_1}tu & \frac{w_4}{w_2}\frac{v}{u} & \frac{w_4}{w_3}\frac{1}{tv} & 1 \end{pmatrix} \tag{34}$$

5.3.3. *Filtering?* Following the above procedure, the next issue is to determine whether AHP filters out the noise. For AHP, this filtering occurs iff AHP assigns the same weights to vectors $\mathbf{a}$ and $\mathbf{a}'$ (Corollary 2). This can happen.

Theorem 10. *For $N \geq 3$, if the difference between $\mathbf{a}'$ and $\mathbf{a}$ is due to a single $\mathbf{C}_\pi$ (Definition 2), the AHP weights assigned to both vectors agree.*[2] *Thus for $N = 3$, AHP filters out all noise; the weights assigned to $\mathbf{a}'$ and $\mathbf{a}$ always agree.*

In the more general $N \geq 4$ case where noise comes from more than one $\mathbf{C}_\pi$, the AHP weights assigned to $\mathbf{a}'$ and $\mathbf{a}$ need not be the same.

Proof. Because $\mathbf{a}'$ satisfies Equation 2, the $\mathbf{a}$ matrix for $N = 3$ and noise c becomes

$$A = \begin{pmatrix} 1 & \frac{w_1}{w_2}c & \frac{w_1}{c\,w_3} \\ \frac{w_2}{c\,w_1} & 1 & \frac{w_2}{w_3}c \\ \frac{w_3}{w_1}c & \frac{w_3}{c\,w_2} & 1 \end{pmatrix}$$

where $a'_{i,j} = \frac{w_i}{w_j}$ and the w_j values are defined by the eigenvector in the $c = 1$ consistency setting. A direct computation shows for this $\mathbf{w} = (w_1, w_2, w_3)$ that

$$A\,\mathbf{w}^T = (1 + c + \frac{1}{c})\mathbf{w}^T, \tag{35}$$

where "T" represents the transpose. As Equation 35 proves, the main affect of the cyclic "c" value is to change the positive eigenvalue from 3 to $(1 + c + \frac{1}{c})$; the eigenvector that determines AHP weights remains unchanged.

In general, consider the j^{th} row of the matrix corresponding to a $N \geq 3$ value where the noise comes from a single cyclic $x\mathbf{C}_\pi$; let $\mathbf{w}$ be the eigenvector for the consistent $\mathbf{a}'$ representation. If k and i, respectively, immediately precede and follow j in π, then $a_{j,k} = \frac{w_j}{x\,w_k}$ and $a_{j,i} = \frac{x\,w_j}{w_i}$; all other terms in this row are of the $\frac{w_j}{w_m}$ form. A direct computation proves that $\mathbf{w}$ is an eigenvector for the inconsistent data with eigenvalue $(N-2) + x + \frac{1}{x}$.

To see that AHP does not filter out the noise for two or more $\mathbf{C}_\pi$ noise components, compute $B(\mathbf{w}^T)$ for B in Equation 34. Filtering occurs iff $1 + \frac{t}{v} + uv + \frac{1}{tu} = 1 + \frac{v}{t} + \frac{t}{u} + \frac{u}{v} = \frac{1}{uv} + \frac{u}{t} + 1 + tv = tu + \frac{v}{u} + \frac{1}{tv} + 1$. Algebraic computations show that any two of these variables must equal unity and the third can assume any value, which reduces to the setting of one $\mathbf{C}_\pi$. (Notice from Equation 34 how each permutation provides a $\frac{1}{u}$ and u multiple in each row, but differences in permutations combine their multiples in different ways in different rows. A general proof would show that if $\mathbf{w}$ times the j^{th} row has the same scalar

[2]While this result and motivation differ significantly from that in DeTurck [3], reference [3] examines interesting related concerns.

times w_j for each j, then the permutations agree. The simpler direct computation suffices for our purposes.) □

5.3.4. *An appropriate, consistent representative.* In general, then, AHP cannot filter out the cyclic noise. This assertion should be of no surprise to AHP advocates with their objective of replacing an inconsistent **a** with a "more" consistent one.[3] Ways adopted to replace inconsistency with consistency include having a subject re-evaluate responses or searching for an appropriate mathematical substitute.

With no other available information, a natural mathematical choice is to find an $\mathbf{a}' \in \mathcal{SC}_3$ that is closest to **a**. For $N = 3$ and letting $x = a_{1,2}$, $y = a_{1,3}$, $z = a_{2,3}$, this reduces to the Lagrange multiplier problem of finding such an $\mathbf{a}'$ that satisfies the $xz - y = 0$ constraint. Unfortunately, the answer need not be unique; $\mathbf{a} = (5, 5, 5)$, for instance, has the two choices: $\mathbf{a}'_1 = (3, 6, 2)$ and $\mathbf{a}'_2 = (2, 6, 3)$ where each is distance $\sqrt{14}$ from **a**. (The weights associated with $\mathbf{a}'_1$ are $w_1 = 2$, $w_2 = 1, w_3 = \frac{1}{3}$.) Approaches of this type are based on the tacit assumption that no information exists about the "error behavior" structure; that is, it is not known how an $\mathbf{a}'$ is converted into an **a**.

Now that we do understand the AHP error structure, the natural approach is to use this formulation to create a modified version of AHP: one where Theorem 10 holds for *all* **a**. Namely the modified version *does* filter out the noise introduced by c values and *does* find the unique, associated $\mathbf{a}' \in \mathcal{SC}_3$. Finding $\mathbf{a}'$ is simple; illustrating with the example problem of $\mathbf{a} = (5, 5, 5)$, it follows (Equation 30) that

$$a'_{1,2} = \frac{5}{c}, \quad a'_{1,3} = 5c, \quad a'_{2,3} = \frac{5}{c},$$

which, with the consistency Equation 2, means that $\frac{25}{c^2} = 5c$, or $c = 5^{\frac{1}{3}}$. Thus the unique

$$\mathbf{a}' = (5^{\frac{2}{3}}, 5^{\frac{4}{3}}, 5^{\frac{2}{3}}), \qquad \text{with weights } w_1 = 5^{\frac{2}{3}},\ w_2 = 1,\ w_3 = 5^{-\frac{2}{3}} \tag{36}$$

Although $\mathbf{a}'$ is farther from $(5, 5, 5)$ than $(2, 6, 3)$ or $(3, 6, 2)$, it more accurately reflects the *structure* of AHP by eliminating only the cyclic noise terms. Moreover, this method faithfully retains the consistent $\mathbf{a}'$ part of the data; "closest point to **a**" approaches do not.

Adding support to this modified AHP approach is Theorem 10, which (for $N = 3$) asserts that the weights assigned to $\mathbf{a} = (5, 5, 5)$ and to $\mathbf{a}' = (5^{\frac{2}{3}}, 5^{\frac{4}{3}}, 5^{\frac{2}{3}})$ must agree. The natural approach to find a consistent partner for inconsistent data, then, is to use Corollary 2 general representation of **a** to eliminate the noise and find the associated $\mathbf{a}'$.

Example: To find an appropriate $\mathbf{a}' \in \mathcal{SC}_4$ for $\mathbf{a} = (4,\ 6,\ 1.5,\ 3,\ 3,\ 8) \notin \mathcal{SC}_4$, the Equation 32 coordinate system (with $x = c_{2,3}$, $y = c_{2,4}$, $z = c_{3,4}$) defines the expressions

$$a'_{1,2} = \frac{4}{xy},\ a'_{1,3} = 6\frac{x}{z},\ a'_{1,4} = 1.5yz,\ a'_{2,3} = \frac{3}{x},\ a'_{2,4} = \frac{3}{y},\ a'_{3,4} = \frac{8}{z}.$$

Substituting these $a'_{i,j}$ values into the three consistency constraints creates three equations with three unknowns: $a'_{1,2}a'_{2,3} = a'_{1,3}$ or $(\frac{4}{xy})(\frac{3}{x}) = 6\frac{x}{z}$; $a'_{1,2}a'_{2,4} = a'_{1,4}$ or $(\frac{4}{xy})(\frac{3}{y}) = 1.5yz$;

[3] But such advocates may be surprised by Theorem 10, which identifies where AHP does accomplish the appropriate filtering.

and $a'_{1,3}a'_{3,4} = a'_{1,4}$ or $(6\frac{x}{z})(\frac{8}{z}) = 1.5yz$. ($F$ converts this system into easier solved linear equations.) The solution $x = 2$, $y = 1$, $z = 4$ leads to $\mathbf{a}' = (2, 3, 6, 1.5, 3, 2)$. The associated weights, then, are multiples of $\mathbf{w} = (2, 1, \frac{2}{3}, \frac{1}{3})$.

5.3.5. *A simple way to find AHP weights.* As shown, the natural objective of converting AHP into a new system that filters out the cyclic noise always can be done. Moreover, a much simpler way to determine final answers is to use Theorem 7 and Equation 25.

To illustrate with the earlier $N = 3$ example of $\mathbf{a} = (5, 5, 5)$, the associated F image is $\mathbf{d} = (\ln(5), \ln(5), \ln(5))$. Thus

$$\bar{b}_1 = \frac{1}{2}[\ln(5) + \ln(5)] = \ln(5), \bar{b}_2 = \frac{1}{2}[-\ln(5) + \ln(5)] = 0, \bar{b}_3 = \frac{1}{2}[-\ln(5) - \ln(5)] = -\ln(5).$$

According to Equations 25 and 26 with $N = 3$, all AHP weights are common multiples of

$$w_1 = (exp(\ln(5)))^{\frac{2}{3}} = 5^{\frac{2}{3}},\ w_2 = 1,\ w_3 = 5^{-\frac{2}{3}},$$

which are precisely those in Equation 36. As it must be expected, these weights differ from what would emerge by using the "closest" $(3, 6, 2)$ or $(2, 6, 3)$ because "closest" approaches need not be compatible with the AHP structure.

As another illustration, the above $\mathbf{a} = (4,\ 6,\ 1.5,\ 3,\ 3,\ 8)$ values define $d_{1,2} = \ln(4)$, $d_{1,3} = \ln(6)$, $d_{1,4} = \ln(1.5)$, $d_{2,3} = \ln(3)$, $d_{2,4} = \ln(3)$, $d_{3,4} = \ln(8)$. Thus

$$\begin{aligned} \bar{b}_1 &= \tfrac{1}{2}[\ln(4) + \ln(6) + \ln(1.5)] = \ln(6), & \bar{b}_2 &= \tfrac{1}{2}[-\ln(4) + \ln(3) + \ln(3)] = \ln(\tfrac{3}{2}), \\ \bar{b}_3 &= \tfrac{1}{2}[-\ln(6) - \ln(3) + \ln(8)] = \ln(\tfrac{2}{3}), & \bar{b}_4 &= \tfrac{1}{2}[-\ln(1.5) - \ln(3) - \ln(8)] = \ln(\tfrac{1}{6}) \end{aligned}$$

According to Equation 26 (with $N = 4$) the AHP weights are a common multiple of $w_1 = \sqrt{6}$, $w_2 = \sqrt{\frac{3}{2}}$, $w_3 = \sqrt{\frac{2}{3}}$, $w_4 = \sqrt{\frac{1}{6}}$. The multiple $\frac{1}{\sqrt{\frac{3}{2}}}$ (so that $w_2 = 1$) yields the $w_1 = 2$, $w_2 = 1$, $w_3 = \frac{2}{3}, w_4 = \frac{1}{3}$ choice derived earlier. In other words, by discovering the appropriate AHP structure, a modified version of AHP can be created to handle "inconsistent" data. The approach uses F to convert the data into a BAR setting; Theorem 7 and Equation 25 (the BAR outcome) eliminates all noise and computes appropriate AHP weights.

6. Summary

It is interesting how a paired comparison data space can be decomposed into data aspects that satisfy a strong transitivity condition and data combinations that create cyclic effects. What adds to the interest is that nothing goes wrong on the $\mathcal{ST}_N$ subspace; on this subspace, all paired comparison rules can be expected to agree. An immediate consequence is that all disagreements among different rules, all paradoxical kinds of behavior, are caused by the ways in which rules react to the cyclic data terms from $\mathcal{C}_N$. This reality creates a procedure for analyzing paired comparison rules. Namely, compare and analyze the rule's outcome for $\mathbf{d} \in \mathcal{ST}_N$ with that of $\mathbf{d} + \mathbf{d}_1$ where $\mathbf{d}_1 \in \mathcal{C}_N$.

This procedure quickly extracts BAR rule properties: BAR outcomes have the positive features that they strictly depend upon the $\mathcal{ST}_N$ components of data and they cancel the cyclic $\mathcal{C}_N$ terms. Moreover, the BAR difference for any specified pair strictly depends on the strongly transitive data components for these two alternatives; what happens with other alternatives is irrelevant. ((This statement does *not* mean that BAR satisfies "Independence of Irrelevant Alternatives" from Arrow's seminal theorem [1]. This feature (to be discussed elsewhere) remains subtly affected by $\mathcal{C}_N$ terms.)

This procedure is further demonstrated by answering natural questions about AHP. As shown, the multiplicative consistency condition (Equation 2) can be equated with the strong transitivity condition; inconsistent terms are uniquely represented in terms of a consistent entry, which is then distorted by multiplicative cyclic effects. The procedure also indicates how to slightly modify AHP in order to filter out the noise embedded in the inconsistent data. Of surprise is the quick, elementary way to compute the modified AHP weights.

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