RESEARCH STATEMENT

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1. INTRODUCTION

In topology, often the primary goal is classification, be it spaces, their structures (triangulations, that of a smooth manifold) or their maps. Often, it is not clear at the beginning which structures one should keep track of or which structures are equivalent. It took nearly sixty years for the study of compact surfaces to reach its full conclusion. The study of 3-manifolds is rapidly progressing, thanks to pioneering ideas of Bill Thurston. Topological simply connected 4-manifolds were classified by Freedman in the late '70s, proving that topological 4-manifolds obeyed the same algebraic master as those in high dimensions. Smooth 4-manifolds however have the same geometric complexity as 3-manifolds, but without the dominance of the fundamental group.

Donaldson's work in the '80s gave indication that geometric structures might play a role in understanding smooth 4-manifolds and the evidence has grown substantially since then following of the Seiberg-Witten invariant and the work of Taubes connecting this to the notion of a symplectic structure.

Symplectic topology developed alongside this with methods slowly approaching the topological cut-and-paste techniques used in smooth 4-manifold topology: finding the means to construct a symplectic manifold by gluing together local pieces. Historically, symplectic manifolds were found whole, say as algebraic varieties. Indeed, Gompf and Mrowka used such cut-and-paste techniques to show that the world of symplectic 4-manifolds was much bigger than the complex world, bringing symplectic topology to the forefront of 4-dimensional topology.

The idea of a symplectic manifold generalizes a natural 2-form on the phase space of a point particle in $\mathbb{R}^n$ in classical mechanics. Each point defines to two pieces data, its position and momentum vectors, giving coordinates on $\mathbb{R}^{2n}$. Naturally associated to this physical system is a 2-form $\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$, where p_i is a coordinate in $\mathbb{R}^n$ and q_i its corresponding momentum direction. This symplectic form naturally decomposes the space into 2-dimensional factors, those spanned by p_i and q_i for a fixed i. Symplectic topology is the study of more general manifolds admitting this type of 2-form, an ω which is both closed ($d\omega = 0$) and non-degenerate ($\omega \wedge \cdots \wedge \omega > 0$), though locally all such symplectic manifolds look like our model on $\mathbb{R}^{2n}$. The easiest example is a surface with an area form or more generally complex surfaces or algebraic varieties.

First Gromov and Eliashberg and later Gompf showed that symplectic manifolds are both geometrically rigid and incredibly general. In the drive to understand the 4-dimensional Poincaré conjecture, Symplectic topology has been extremely useful. The known small exotic 4-manifolds (small in this case means those 4-manifolds homeomorphic to $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ for $k \leq 9$) admit symplectic structures for each smooth structure and certain smooth manifolds admit only one.

This search for cut-and-paste methods led to the study of the induced structures on boundary 3-manifolds. In nicest cases, 3-manifolds in symplectic manifolds admit natural *contact structures*. One way to view this is to think of symplectic manifolds as generalizing complex manifolds. In particular, Gromov proved that on any symplectic manifold W there exists a bundle automorphism J (an *almost complex structure*) of the tangent bundle TW which has square equal to $-Id$, just as the complex number $\sqrt{-1}$ induces a skew-involution of the tangent bundle of a complex manifold. We can think of the contact structure on $M = \partial W$ as the subspace of TM which is preserve by J. Specializing to the case we care about, W has dimension 4, M 3 and this invariant field is a 2-dimensional subbundle of TM. This bundle is most often described as the kernel of a 1-form α on M which satisfies $\alpha \wedge d\alpha > 0$, analogous to the non-degeneracy condition of the symplectic form. It is here that contact topology in dimension 3 truly started, by looking at certain plane fields occurring as the boundary of symplectic 4-manifolds.

Our fundamental understanding of contact structures changed with the 2002 result of Giroux, showing that contact structures are equivalent to *open book decompositions*. As a topological object, an open book decomposition refers just to the fibration structure on the complement of a fibered link. Giroux's theorem connects the worlds of smooth and contact topology in a way that parallels a similar (though weaker) relationship given by Donaldson between symplectic manifolds and Lefschetz fibrations, a different type of singular fibration on a 4-dimensional manifold which generalizes elliptic fibrations.

This correspondence is where I apply most of my efforts and has proven to be a powerful tool in both low-dimensional topology and symplectic topology. The correspondence is fundamental in the proof of Property P for knots (that Dehn surgery on only the unknot can give S^3) and in the definition of the Oszvath-Szabo contact invariant in Heegaard Floer homology. Its relation to Lefschetz fibrations can be used to both classify symplectic fillings (generalizing results of Y. Eliashberg, D. McDuff, A. Stipsicz and more recently P. Lisca) and to construct distinct fillings (useful for cut-and-paste modifications of symplectic manifolds, such as *rational blowdown*, for example).

2. Previous Research

Fillings and planar spinal open books. A result of C. Wendl using holomorphic curves says that every symplectic filling of an open book whose page has genus 0 comes from a Lefschetz fibration which restricts at the boundary to the same open book. This means that every symplectic filling comes from a factorization of the monodromy into a product of right-handed Dehn twists. O. Plamenevskaya and I used this theorem to extend work of Eliashberg, of McDuff and of Lisca on the symplectic fillings of the standard contact structure on S^3, $L(4,1)$ and $L(p,1)$, resp. Each of these previous classifications of symplectic fillings were for the unique tight contact structure lifting to the tight contact structure on S^3. Our theorem works for all tight contact structures, save McDuff's example of $L(4,1)$, which has two different fillings.

Theorem (Plamenevskaya-V.)**.** *Every tight contact structure on the lens space $L(p,1)$ with $p \neq 4$ (and there are p-1 of them) admits a unique symplectic filling, up to deformation and blow-up.*

Plamenevskaya and I use the abelianization of the planar mapping class group (along with some heavy topological results) to prove the previous theorem. We also use Wendl's theorem to provide obstructions for planar open books to admit a symplectic filling. We give (previously known) examples of tight, planar contact manifolds (with non-vanishing Ozsvath-Szabo contact invariant) which are not fillable. I believe that as our understanding of the planar mapping class group grows, these can be significantly extended. There is still a lot of interesting work to be done.

With analysis done by Wendl and S. Lisi, we have worked to extend Wendl's methods by thinking about the appropriate objects which generalize open book decompositions to the induced structure on the boundary of a Lefschetz fibration over a non-disk base. Such decompositions we call *spinal open books*. A spinal open book is very much like a standard open book except that you allow the binding to be the product of S^1 and any (likely disconnected) surface. Wendl and Lisi and I, in forthcoming paper work, have proved the spinal analogue of Wendl's result.

Theorem (Wendl-Lisi-V.)**.** *Any strong filling of a spinal open book $\mathfrak{ob}$ with a planar component of the page comes from a Lefschetz fibration whose boundary is $\mathfrak{ob}$. Any minimal such filling is Stein.*

This characterization gives two interesting obstructions to a spinal open book supporting a fillable contact structure. The *spine* of a spinal open book is our term for the new binding. The first filling obstruction comes from the topology of the spine, and in particular says that every component of the spine must be homeomorphic. This condition is automatically satisfied for honest open books and is one reason spinal open books are useful. The second condition is as in my work with Plamenevskaya and comes from analyzing what monodromies can be realized as the boundaries of Lefschetz fibrations. A combination of the two obstructions gives the following classification of fillable contact structures on the 3-manifolds $Y(g,k)$, the circle bundles of Euler number k over a surface of genus g. The manifolds $Y(g,k)$ were some of the first to have their contact structures classified, done by K. Honda. These give many examples of planar spinal open books which cannot admit planar honest open books and so Wendl's previous theorem does not apply.

Theorem (Wendl-Lisi-V.). *An S^1-invariant contact structure on $Y(g,k)$ is fillable if and only if either*

(1) *the convex base surface S has connected positive and negative regions S_+ and S_- with $S_+ = S_-$ and $k \geq 0$ or*
(2) *$k < 0$ and either S^+ or S^- is empty,*

which completes the study of strong-fillability for contact structures on $Y(g,k)$.

Cables, rational open books. Open book decompositions have allowed researchers to apply many tools from low dimensional topology to the study of contact structures. K. Baker, J. Etnyre, and I [1] studied *rational open books* and the behavior of contact structures under cabling, with many interesting results. Rational open books are just open books whose binding link represents a non-trivial torsion class in homology. In this case, each page of the open book wraps many times around each component of the binding link as an (s,r)-cable. While the integral case is straight forward, cabling rational open books has unexpected behavior which is highly interesting and which we do not yet fully understand. Positive cabling does not change (the isotopy type of) the supported contact structure. Most negative cables (those of honest open books, for example), though, are overtwisted, while some can be tight.

Theorem (Baker-Etnyre-V.). *A positive (p,q) cable of an (r,s) open book supports the same contact structure. A negative (p,q) cable is overtwisted with a possible exception if q/p is connected to r/s by an arc in the Farey tessellation. There exists negative cables of rational open books which support tight contact structures.*

We also show that for most cables, the open book decomposition on the cable can be made by adding Hopf bands to the original open book, but for certain cables, this is not true. We use this to answer a question posed by Honda-Kazez-Matić [10], informed by Giroux, regarding factorizations of monodromies of Stein fillable contact structures - some open books compatible with Stein fillable contact structures have no factorization of their monodromy as a product of positive Dehn twists.

Theorem (Baker-Etnyre-V.). *There exists Stein fillable contact structures which are compatible with open book decompositions whose monodromy cannot be written as a product of positive Dehn twists.*

We show do show, however, that there is some monoid in the mapping class group characterizing those monodromies compatible with Stein fillable contact structures. In fact, there are monoids corresponding to all classes of contact structures which are closed under Legendrian surgery - there are monoids for each type of symplectic fillability, for non-vanishing Heegaard Floer invariant (a fact first proved by Baldwin), as well as the hypertight condition of Colin and Honda. Characterizing these monoids and their relationships to the previously known monoids, $Veer$ and $Dehn^+$, would be very interesting and potentially powerful result which is discussed in the next section.

Theorem (Baker-Etnyre-V.). *For every surface S, there is a monoid in the mapping class group containing exactly those monodromies compatible with Stein fillable contact structures. Such monoids characterize weakly and strongly fillable contact structures, as well as those whose Ozsváth-Szabó contact invariant is non-trivial.*

Fibered Links. While the above application illustrates how one can use topological methods and the study of knots and links to understand contact manifolds, it is also possible (and highly useful) to go the other way, as well. A classical example of this is the Bennequin inequality which compares the *self-linking number* of a link to its Seifert and slice genera, which was used effectively to determine both of these invariants for many small knots. The Bennequin bound holds in all manifolds and is quite powerful (for example, it characterizes tight contact structures and distinguishes exotic 4-manifolds) and not very well understood. Much effort has been devoted to understanding the classification of Legendrian and transverse links and the Bennequin bound. Etnyre and I ([4]) prove that for fibered links, the Bennequin bound is sharp if and only if the fibered link supports the contact structure (up to Giroux torsion). To do this, Etnyre and I dive into the dicey world of convex surfaces with transversal boundary (cf. [8], [7] for the better behaved Legendrian and closed cases).

Theorem (Etnyre-V.). *Let L be a transverse, fibered link in a tight contact 3-manifold (M, ξ). The Bennequin bound is sharp for L if an only if ξ is supported by an open book with binding L (up to Giroux torsion).*

This fact has many interesting consequences. On S^3, there are many bounds on the self-linking number that generalize and improve the Bennequin bound. Rudolph pushed the bound to the smooth 4-genus and there are additional bounds coming from HOMFLYPT polynomial as well as the knot homologies of Khovanov and Ozsváth and Szabó. Sharpness of the Bennequin bound implies that all of these invariants are equal. Moreover, work of Bennequin and Wrinkle closely ties the study of transverse links in S^3 to that of braids. Etnyre and I apply the techniques above to deduce new results for braids. In addition, this classifies all transverse representatives of these links, in S^3 or not, with maximal self-linking number.

Classification. The fundamental question in the contact topology of 3-manifolds is classification. What are the tight contact structures on an oriented 3-manifold? Which satisfy the various fillability criteria? P. Ghiggini and I answer an a long-standing open question about the classification of tight contact structures on the Brieskorn homology spheres $-\Sigma(2,3,6n-1)$. Lisca and Matić [12] first studied these contact manifolds using gauge theory and were able to find $n(n-1)/2$ tight contact structures. Unfortunately, unlike previous classifications, they were unable to show that all the contact structures were distinct. (In earlier work, they were able to show that some were distinct using their Stein fillings, though as we see, not all admit a Stein filling.) More recent progress was made by Wu [14] and Ghiggini [6]. Using open book decompositions and Heegaard Floer homology, Ghiggini and I complete the classification of contact structures on these manifolds.

Theorem (Ghiggini-V.). *There are exactly n(n-1)/2 tight contact structures on the Brieskorn spheres $-\Sigma(2,3,6n-1)$. They are distinguished by their Ozváth-Szabó contact invariants.*

Symplectic and smooth 4-manifolds. Strongly paralleling the notion of contact manifolds as the boundary of symplectic manifolds, open book decompositions are often the boundaries of Lefschetz fibrations. T. Mark and I use monodromy factorizations to construct symplectic rational balls filling a family contact manifolds, giving explicit pictures of the rational blowdown procedure [2], [5] for symplectic 4-manifolds. Rational blowdown constructs exotic 4-manifolds by removing neighborhoods of nicely immersed surfaces and replacing them with rational balls. The description obtained by Mark and I illuminates the construction for symplectic manifolds in terms of Lefschetz fibrations and H. Endo provides some very nice examples of Lefschetz fibrations where applying these substitutions creates a non-standard manifold. This work is related to a conjecture of Kollár [11] and the question of which Seifert manifolds bound contractible 4-manifolds and has applications to the study of algebraic surfaces.

Generalizing the idea of a rational ball, a 4-manifold X whose boundary is a graph manifold M_G is *rationally minimal* if the rank of $H_1(X)$ is as small as possible, as determined by the weighted graph G.

Question. *Which graph manifolds M_G bound rationally minimal Stein manifolds, i.e. a manifold whose first homology has rank $b_1(G) + \sum 2g(v)$, where $b_1(G)$ is the rank of the first homology of the underlying graph G and $g(v)$ is the genus of the vertex v.*

Such Stein fillings would be useful in cut-and-paste constructions of symplectic manifolds. These methods have been very successful in finding ever smaller exotic 4-manifolds with the hope of understanding the smooth 4-dimensional Poincaré conjecture.

3. Future Directions

Page genus and monodromy monoids. One very interesting direction for future research is to understand the two main geometric invariants of contact structures arising from open book decompositions: the minimal page (or support) genus and the set of monodromies. The support genus is a powerful invariant (it detects tightness) but little is known about it in general. Though every 3-manifold admits some planar open book, there are obstructions, coming from symplectic topology [3] and Heegaard Floer homology [13], for a contact structure to be compatible with a planar open book decomposition. Nothing is known for positive genus, however. In fact, it is even possible (though highly unlikely) that every contact structure admits a compatible genus 1 open book!

But, an open book has a return map or monodromy as well, which (along with my work discussed earlier) gives us to the second invariant, the monodromy monoids. This idea isn't without precedent. Honda Kazez and Matić determine tightness of a contact structure in terms of the *right-veering monoid* and the compatible monodromies. Moreover, the monodromy of an open book actually completely determines whether the compatible contact structure is symplectically fillable (and whether such fillings are weak, strong, Stein, exact and so on) and also whether the Ozsváth-Szabó invariant is non-zero because for each of these properties, there is a monoid in the mapping class group of the page containing all monodromies compatible with such contact structures. While progress has been made in the genus zero and one cases, characterizing these monoids would be a monumental feat.

There is also indication that through branched covers, there is a relationship between the various fillability properties and known knot invariants of the branch locus. For example, L. Watson and I have found that every 3-braid which satisfies the self-linking bound for the Khovanov s-invariant has an open book supporting a tight contact structure as its two-fold branched cover. We believe that there are still many other interesting relationships between braids and their branched covers yet to be found.

Legendrian and transverse knots. The classification problem for general contact manifolds is quite complex. When M is a contact manifold with convex boundary, then the classification program can be simplified somewhat. When M has torus boundary, the the classification of tight contact structures is closely related to the classification of Legendrian and transverse knots on a Dehn filling of M.

There are many reasons to look at this setting: the close connection between sutured manifold decompositions and contact structures [9] being no small part, as well as the growing understanding of the contact invariant in sutured Floer homology. Results here would generalize my work in [4] with Etnyre and likely have similar applications to the study of braids. Though a complete classification is the goal, the natural starting point is those contact structures with a maximal Euler class, essentially the maximal self-linking condition I used in the fibered case. The right answer here would further generalize Giroux's compatibility and have applications to the study of broken Lefschetz fibrations as well as shed light on the understanding of contact handle decompositions.

Open book decompositions of 5-manifolds. Unlike dimension 3, not every 5-manifold admits a contact structure. To begin, there is a topological obstruction to admitting a coorientable codimension-1 distribution (an *almost contact structure*). It is unknown, though, whether every 5-manifold admitting an almost contact structure also admits a single contact structure, and one would like to know that there is a contact structure for every almost contact structure. While open book decompositions are defined for all manifolds, there are additional assumptions needed in higher dimensions for this to give a contact structure. First, one needs a particularly nice symplectic manifold as the page of the open book but second, one needs the monodromy to respect the symplectic structure on the page. The second condition is extremely difficult to guarantee in general. However, just as Wendl's theorem allowed Plamenevskaya and I to completely characterize all Lefschetz fibrations filling a certain contact manifold, it also allows one to better understand the symplectic mapping class group of the Stein fillings. To construct a contact structure, then, one needs to find an open book decomposition with a particularly nice Stein page; one whose boundary is compatible with a planar open book whose factorizations are particularly well behaved. I will work to find such open books on all almost contact 5-manifolds.

Factorizations in the pure braid group/planar mapping class group. My work with Plamenevskaya uses the abelianization of the planar mapping class group to extract information about possible positive factorizations of the monodromy of an open book. We needed to apply some heavy additional topological results to get our theorem, but these means should not be necessary. The planar mapping class group though has a nice presentation with essentially one relation: the lantern relation. With a better understanding of the planar mapping class group, one should be able to completely determine all positive factorizations of a monodromy and hence all symplectic fillings. (A nice generalization of a Garside structure would do this, for example.) An intermediate step, though, would be to look for a nonabelian representation that detects more of the structure than the abelianization but without tackling the complexity of the full group. I will work to find other interesting representations of the planar mapping class group and their applications to symplectic topology.

REFERENCES

[1] Kenneth L. Baker, John B. Etnyre, and Jeremy Van Horn-Morris. Cabling, rational open book decompositions and contact structures. In preparation.

[2] Andrew J. Casson and John L. Harer. Some homology lens spaces which bound rational homology balls. *Pacific J. Math.*, 96(1):23–36, 1981.

[3] John B. Etnyre. Planar open book decompositions and contact structures. *Int. Math. Res. Not.*, (79):4255–4267, 2004.

[4] John B. Etnyre and Jeremy Van Horn-Morris. Fibered Transverse Knots and the Bennequin Bound, 2008.

[5] Ronald Fintushel and Ronald J. Stern. Rational blowdowns of smooth 4-manifolds. *J. Differential Geom.*, 46(2):181–235, 1997.

[6] Paolo Ghiggini. Strongly fillable contact 3-manifolds without Stein fillings. *Geom. Topol.*, 9:1677–1687 (electronic), 2005.

[7] Emmanuel Giroux. Convexité en topologie de contact. *Comment. Math. Helv.*, 66(4):637–677, 1991.

[8] Ko Honda. On the classification of tight contact structures. I. *Geom. Topol.*, 4:309–368 (electronic), 2000.

[9] Ko Honda, William H. Kazez, and Gordana Matić. Convex decomposition theory. *Int. Math. Res. Not.*, (2):55–88, 2002.

[10] Ko Honda, William H. Kazez, and Gordana Matić. Right-veering diffeomorphisms of compact surfaces with boundary. *Invent. Math.*, 169(2):427–449, 2007.

[11] János Kollár. Is there a topological Bogomolov-Miyaoka-Yau inequality? *Pure Appl. Math. Q.*, 4(2, part 1):203–236, 2008.

[12] Paolo Lisca and Gordana Matić. Tight contact structures and Seiberg-Witten invariants. *Invent. Math.*, 129(3):509–525, 1997.

[13] Peter Ozsváth, András Stipsicz, and Zoltán Szabó. Planar open books and Floer homology. *Int. Math. Res. Not.*, (54):3385–3401, 2005.

[14] Hao Wu. On legendrian surgeries. *Math. Res. Lett.*, 14(3):513–530, 2007.