# Personal research statements and open problems

Workshop
Mathematical aspects of physics with non-self-adjoint operators

June 8–12, 2015

## 1 I. Alexandrova

My research interests lie in the areas of scattering theory, semi-classical and microlocal analysis, magnetic Hamiltonians, and the Aharonov-Bohm effect.

## 2 Y. Almog

We consider the eigenspace of $\mathcal{P} : D \to L^2(\Omega)$, where

$$\mathcal{P} = -\sum_{k=1}^{d} e^{2i\alpha_k}(\partial_k - iA_k)^2 + V$$

In the above $A = (A_1, \dots, A_d)$ is a smooth magnetic potential, and $V$ is a complex potential. The domain is a smooth unbounded subset of $\mathbb{R}_d$, and $D$ is a subset of $H_0^1(\Omega)$. We then apply the results to the linearized Ginzburg-Landau operator in a half-space.

Joint work with Bernard Helffer.

## 3 S. Bögli

Given an unbounded non-selfadjoint linear operator $T$, I am particularly interested in

1) how to reliably compute the spectrum of $T$ by approximating $T$ by a sequence of simpler linear operators $T_n$ and computing their eigenvalues (in particular, sufficient conditions for spectral exactness, how to locate spurious eigenvalues, perturbation theory);

2) pseudospectra of $T$ (in particular, open sets of constant resolvent norm, perturbation theory for pseudospectra);

3) applications in mathematical physics (in particular, magnetohydrodynamic dynamo models, Schrödinger operators with complex potentials).

Results related to 1) and 3) can be found in my PhD thesis (University of Bern, 2014); for results regarding 2) see the joint paper with P. Siegl [BS14].

## 4 L. Boulton

### 4.1 Schauder bases of periodic functions and multipliers

A family of highly non-self-adjoint operators characterised by Dirichlet multipliers arises naturally in the study of basis properties of dilated functions on a segment. The theory is well developed in the $L_2$ (Hilbert) space setting. During the workshop I will be interested in discussing possible extensions of this theory to the case of the Banach spaces $L_s$ for $s \neq 2$.

Natural open problems. The topic might be of interest to Mityagin, Shkalikov, Shubov and others.

**Q1:** Determine the spectrum of Dirichlet multipliers in $L_s$.

**Q2:** Bounds for the norm of a Dirichlet multiplier in terms of its associated Dirichlet series/symbol. Impact on basis properties of dilated functions.

**Q3:** Schauder non-basisness theorems for general families of dilated functions and decay of the Fourier coefficients of the generator.

**Q4:** Basisness and applications to non-orthogonal projection methods for the $p$-Schrödinger parabolic time-evolution initial value problem.

### 4.2 Numerical approximation of rigourous enclosures for the spectrum of J-self-adjoint operators

The aim of this theme is to device strategies for computing rigourous (hopefully sharp) bounds/enclosures for the spectrum of J-self-adjoint operators by means of projected space methods. The theory of computation for spectra of self-adjoint operators is classical and well developed. J-self-adjoint operators share many properties with their self-adjoint counterpart. During the workshop I will be interested in discussing to what extent general strategies for numerically estimating spectra of J-self-adjoint operators can be devised.

**Q1:** Computation of rigourous rough enclosures for J-self-adjoint operators, taking into account the structure of the conjugation J into the projection scheme.

**Q2:** Computation of sharp (numerically relevant) enclosures in specific or generic cases.

**Q3:** Impact in the study of evolution problems for J-self-adjoint operators.

## 5 A. Boumenir

### 5.1 Nonselfadjoint Inverse Problems

We are interested in identifying a nonself adjoint operator associated with an evolution equation (parabolic or hyperbolic) through "observations" of the solution as time evolves. Thus for example in a certain Hilbert space we have

$$u'(t) = Au(t) \quad \text{ and } u(0) = f \tag{1}$$

where, for simplicity, we assume that

$$A = L + B$$

with $L$ is a given (known) self-adjoint operator with "nice properties" while $B$ is an unknown non selfadjoint perturbation. For example $Ay(x) = y''(x) - q(x)y(x)$ or $Au = \Delta u - q(x)u$ with $\operatorname{Im} q(x) \neq 0$. We assume that we can observe the solution through a functional $< ., \ g >$ say

$$\omega(t) =< u(t), \ g > .$$

For example if $u(x,t)$ is the solution of a heat equation, where $x \in \Omega \subset \mathbb{R}^n$, and $p \in \partial\Omega$, then $\omega(t) = u(p,t)$ (temperature) or $\omega(t) = \partial_n u(p,t)$ (heat transfer) are usual observations/readings of the solution on the boundary. Thus we want to recover $A$ or at least its spectrum $\sigma_A = \{\lambda_n\} \subset \mathbb{C}$ from the observation mapping

$$u(0) \to \omega(t).$$

To do so, although we do NOT know $A$, we assume that it has a discrete spectrum $\{\lambda_n\} \subset \mathbb{C}$, and in general $\operatorname{Im} \lambda_n \to 0$ as $n \to \infty$, while $\operatorname{Re} \lambda_n \to -\infty$. If we denote its eigenfunctions by $\varphi_{n,0}$ and its associated eigenfunctions (roots) by $\varphi_{n,\nu}$ for $\nu = 1, ..., m_n - 1$, where $m_n$ is the multiplicity of the eigenvalue $\lambda_n$, then we can write a formal solution to the evolution equation

$$u(t) = \sum_{n \geq 1} e^{\lambda_n t} \sum_{\nu=0}^{m_n - 1} c_{n\nu}(f) \, p_{n\nu}(t) \varphi_{n\nu} \tag{2}$$

where the Fourier coefficients are $c_{n\nu}\,(f) =< f,\ \psi_{n\nu} >$ and $\{\psi_{n\nu}\}$ is the biorthogonal system to $\{\varphi_{n,\nu}\}$. Here $p_{n\nu}$ are polynomials generated by the multiplicity of the eigenvalue $\lambda_n$. The observation then is given by

$$\omega(t) = \sum_{n\geq 1} e^{\lambda_n t} \sum_{\nu=0}^{m_n-1} c_{n\nu}\,(f)\, p_{n\nu}(t) < \varphi_{n\nu}, g > . \tag{3}$$

In the best case, when **all** $c_{n\nu}\,(f) \neq 0$ and $< \varphi_{n\nu}, g > \neq 0$ then it is possible to evaluate/extract **all** the $\lambda_n$ from the observation (2).

This raises the following questions:

**Q1:** How do you choose the initial condition $f$, so we can observe all $e^{\lambda_n t}$, that is **all** $c_{n\nu}\,(f) \neq 0$.
We need to know something about the biorthogonal system $\{\psi_{n\nu}\}$.

**Q2:** How do you choose the observation $g$ so **all** $< \varphi_{n\nu}, g > \neq 0$
We need to know something about the root functions $\{\varphi_{n,\nu}\}$.

**Q3:** How smooth is the sum (2)? so we can choose $g$
We need some information on the type of convergence in (2) so (3) holds.

**Q4:** How do we extract the $\lambda_n$ and their multiplicity from a given signal given by (3) in finite time?
When $\lambda_n$ are complex values and the sum *contains* polynomials in $t$, it is much harder than the real case.

**Q5:** Find the best $f$ and $g$ that allow the identification of $A$ by using the smallest number of observations.
Evolution equations are often found in control theory, and for that purpose, we need finite number of observations done in finite time..

# 6 T. J. Christiansen

Mathematically, resonances may serve as a replacement for discrete spectral data for a class of operators with continuous spectrum. Physically, they correspond to decaying waves. Although there are a number of ways to define resonances, one is as the eigenvalues of a certain non-self-adjoint operator.

I am generally interested in questions related to the distribution of resonances. Some earlier work used non-self-adjoint Schrödinger operators in proving results for resonances of self-adjoint Schrödinger operators.

An example of issues particularly related to the conference theme is the following. Consider the Schrödinger operator $-\Delta + V$ on $\mathbb{R}^d$, where the potential $V \in L_0^\infty(\mathbb{R}^d)$. If $V \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$, then if $V$ is non-trivial the Schrödinger operator has infinitely many resonances. However, if $d \geq 2$ there are non-trivial complex-valued potentials $V \in C_c^\infty(\mathbb{R}^d)$ for which the corresponding Schrödinger operator has no resonances. More generally, one can explicitly construct families of isoresonant, compactly supported complex-valued potentials in dimensions at least 2.

**Q1:** Is there some other data related in some way to spectral properties of the operators that distinguish elements (potentials) in these sets?

# 7 M. Chugunova

## 7.1 Computations of the instability index for a non-self-adjoint operators

The stability of steady states is a basic question about the dynamics of any partial differential equation that models the evolution of a physical system.

In order to numerically evaluate the instability index of a given differential operator $A$, its computation should be reduced to a problem of linear algebra. Particularly for problems with periodic boundary conditions, it seems natural to restrict the operator $A$ to a finite-dimensional space of trigonometric polynomials.

**Q1:** Under what conditions the instability index (the total number of unstable eigenvalues) can be computed from the resulting finite dimensional matrix?

One difficulty is that the entries of the infinite matrix corresponding to the differential operator $A$ grow with the row and column index, so that any truncation is not a small perturbation.

If $A$ is a self-adjoint semi-bounded differential operator of even order, then the instability index can be estimated by variational methods, or computed directly from the zeroes of the corresponding Evans function.

Understanding the spectrum of a non-self-adjoint operator is a much harder problem. It is not at all obvious how to restrict the computation of its instability index to a finite-dimensional subspace, or how to even estimate its dimension. Furthermore, the numerical calculation of eigenvalues can be extremely ill-conditioned even in finite dimensions.

# 8 M. Demuth

## 8.1 Spectral radius and operator norm

Let $A$ be a bounded linear operator on a Banach space $X$. Its spectral radius is defined by

$$\operatorname{spr}(A) := \max\{|z| \,:\, z \in \sigma(A)\}.$$

Gelfand proved the classical formula

$$\operatorname{spr}(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}.$$

Obviously, $0 \le \operatorname{spr}(A) \le \|A\|$. The question arises: What is the gap between $\|A\|$ and $\operatorname{spr}(A)$? Introduce the denotation

$$\operatorname{gap}(A) := \|A\| - \operatorname{spr}(A).$$

Questions:

**Q1:** For which class of operators holds

$$\operatorname{gap}(A) > 0 \text{ or } \operatorname{gap}(A) = 0,$$

respectively.

**Q2:** What is the smallest $m \in (0, 1]$, such that

$$\operatorname{spr}(A) \le m\|A\|$$

or

$$\operatorname{gap}(A) \ge (1 - m)\|A\| \quad ?$$

**Example 1.** Let $X = \ell^1(\mathbb{N})$ and $A$ be the weighted shift-operator defined according to the canonical standard basis by the infinite matrix

$$\begin{pmatrix} 0 & & & & & \\ b_1 & 0 & & & & \\ & b_2 & 0 & & & \\ & & b_1 & 0 & & \\ & & & b_2 & 0 & \\ & & & & \ddots & \ddots \end{pmatrix}$$

where $b_1, b_2 > 0$ and $b_1 b_2 = 1$. In this case $\|A\| = \max\{b_1, b_2\}$ and $\sigma(A) = \{z \in \mathbb{C} : |z| \le 1\}$ and therefore $\operatorname{spr}(A) = 1$. Thus

- $\mathrm{gap}(A) = 0$: If $b_1 = b_2 = 1$ then $\|A\| = \mathrm{spr}(A)$.
- $\mathrm{gap}(A) > 0$: If $b_1 \neq b_2$ then $\|A\| > \mathrm{spr}(A)$.

This kind of estimates are useful in the following situation.

Let $K$ be a compact perturbation of $A$.
Study the discrete spectrum of $B := A + K$. We are able to analyse the moments and the number of eigenvalues of $B$ outside a ball of radius $\|A\|$. It is more interesting and also natural to enlarge this region up to the complement of a ball with radius $\mathrm{spr}(A)$.

## 8.2 Estimates for the resolvent near the spectrum

Let $A$ be a linear operator on a Banach space. Let $K$ be a compact perturbation of $A$. The approximation numbers of $K$ are defined by

$$\alpha_N(K) := \inf\{\|K - F\|\,,\, \mathrm{rank}(F) < N\}.$$

We consider only compact operators $K$ with $\lim_{N\to\infty} \alpha_N(K) = 0$.

The objective is to estimate the numbers of eigenvalues of the perturbed operator $B := A + K$ in certain regions of the complex plane.

Let $\Omega_t = \{\lambda \in \mathbb{C}\,,\, |\lambda| > t\}$. Denote $\mathrm{spr}(A) := \max\{|\lambda|\,,\, \lambda \in \sigma(A)\}$ and assume $\mathrm{spr}(A) < t < s$. Denote by $n_B(s)$ the number of eigenvalues of $B$ in $\Omega_s$. In [DHHK15] we obtained

$$n_B(s) \leq \frac{(2e)^{\frac{p}{2}}}{\log\frac{s}{t}} \frac{\sup_{\lambda\in\Omega_t} \|(\lambda - A)^{-1}\|^p}{\left(1 - \alpha_{N+1}(K)\sup_{\lambda\in\Omega_t} \|(\lambda - A)^{-1}\|\right)^p} \sum_{j=1}^{N} (\alpha_{N+1}(K) + \alpha_j(K))^p\,. \tag{4}$$

Here $N$ has to be so large that

$$\alpha_{N+1}(K) \sup_{\lambda\in\Omega_t} \|(\lambda - A)^{-1}\| < 1.$$

The optimal result depends on the behaviour of $\|(\lambda - A)^{-1}\|$ near the spectrum of $A$, i.e. on $\Omega_s$. This is typical for many spectral considerations. It is also related to the pseudospectrum of $A$. For instance if $|\lambda| > \|A\|$ then

$$\|(\lambda - A)^{-1}\| \leq \frac{1}{|\lambda| - \|A\|}$$

and therefore (4) becomes

$$n_B(s) \leq \frac{(2e)^{\frac{p}{2}}}{\log\frac{s}{t}\,(t - (\|A\| + \alpha_{N+1}(K)))^p} \sum_{j=1}^{N} (\alpha_{N+1}(K) + \alpha_j(K))^p\,.$$

In this context one should analyze the following questions:

**Q1:** Classify the operators for which the resolvent is polynomially bounded if $\lambda \to \sigma(A)$?

**Q2:** Classify the operators for which one can find an $M \geq 1$ such that

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{\mathrm{dist}\,(\lambda, \sigma(A))}$$

for all $\lambda \in \mathrm{res}(A)$ or

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda| - \mathrm{spr}(A)}$$

for $|\lambda| > \mathrm{spr}(A)$.

**Remark:** For instance in Hilbert spaces Q1 is true for normal operators with $M = 1$.

# 9 N. J. Dencker

I am interested in pseudospectrum, in particular for semiclassical PDE, and the connection with solvability of PDE.

# 10 M. Embree

Functions of Nonnormal Matrices and the Behavior of Dynamical Systems

Nonnormality raises a series of issues that complicate the analysis of dynamical systems. This talk will explore several of these issues, including some open problems in the literature: Davies's conjecture about approximate diagonalization; Crouzeix's conjecture about the norm of matrix functions; the inverse numerical range problem of Malamud, Uhlig, and Carden; the question of "Do pseudospectra determine behavior", as explored by Ransford, Greenbaum, and Trefethen; and the role of nonnormality in the Lyapunov matrix equation.

We shall also discuss how best to generalize pseudospectra to more sophisticated dynamical systems (differential-algebraic equations, delay differential equations, nonlinear differential equations). In particular, we will show how the most natural generalization of pseudospectra for the generalized eigenvalue problem fails to describe transient growth in differential-algebraic equations, and we will propose a more suitable alternative.

R. L. Carden and M. Embree, Ritz value localization for non-Hermitian matrices, SIAM J. Matrix Analysis Appl., 2012. http://dx.doi.org/10.1137/120872693

M. Crouzeix, Numerical range and functional calculus in Hilbert space, 2007. http://dx.doi.org/10.1016/j.jfa.2006.10.0

E. B. Davies, Approximate diagonalization, SIAM J. Matrix Analysis Appl., 2007. http://dx.doi.org/10.1137/06065990

T. Ransford, On pseudospectra and power growth, SIAM J. Matrix Analysis Appl., 2007 http://dx.doi.org/10.1137/060

L. N. Trefethen and M. Embree. Spectra and Pseudospectra. Princeton, 2005. See especially Chapters 25 and 47.

# 11 F. Gesztesy

Relatively few classes of non-self-adjoint operators have been identified as scalar spectral operators in the sense of Dunford (cf. [DS88]). The case of periodic one-dimensional Schrödinger operators in $L^2(\mathbb{R}; dx)$ with a complex-valued periodic potential in $L^2_{loc}(\mathbb{R})$ is one of the exceptions that was characterized in [GT09] (see also the announcement [GT06]).

Since much of the work on non-self-adjoint one-dimensional Schrödinger operators in the new millennium focuses on aspects (such as, existence of Riesz bases, etc.) of periodic, respectively, antiperiodic Schrödinger operators on a compact interval, it seems natural to branch out a bit and hence look for non-periodic examples of scalar spectral operators, for instance, Schrödinger operators in $L^2(\mathbb{R}; dx)$ with quasi-periodic (and eventually, almost periodic) potentials.

A particular class of Schrödinger operators $H = -(d^2/dx^2) + V$ in $L^2(\mathbb{R}; dx)$ that should permit such a characterization is the class of algebro-geometric, finite-gap KdV potentials, that is, the class of smooth solutions of one (and hence, infinitely many) of the equations defining the stationary (i.e., time-independent) KdV hierarchy. The class of algebro-geometric potentials $V$ is characterized in great detail in [BG05], [GH, Ch. 2], and it is known to generally consist of quasi-periodic potentials. We note that one has a precise characterization for the Green's function for such (generally) non-self-adjoint Schrödinger operators $H$ and a description of their spectra in terms of the ergodic mean value of the inverse of the diagonal Green's function of $H$. This implies that the spectrum of $H$ consists of finitely many simple analytic arcs and one semi-infinite arc in the complex plane. Crossings, as well as confluences of spectral arcs are possible and those instances can be described as well (generally, their existence will be sources for $H$ not being a scalar spectral operator).

Analogous comments apply to Jacobi operators and the stationary Toda hierarchy.

# 12 M. Hansmann

## 12.1 Tensor trick for perturbed operators?

Consider a trace class operator $K$ on a complex Hilbert space $\mathcal{H}$. It is well known that the eigenvalue sequence of $K$ is summable and that we have the inequality

$$\sum_n |\lambda_n(K)| \leq \|K\|_{\mathrm{tr}}. \tag{5}$$

For the sake of argument, let us assume that we don't yet know that (5) holds, but that we are only able to prove that for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) \geq 1$, which diverges for $\varepsilon \to 0$, such that

$$\sum_n |\lambda_n(K)|^{1+\varepsilon} \leq C(\varepsilon)\|K\|_{\mathrm{tr}}. \tag{6}$$

Then a simple way to obtain estimate (5) from estimate (6) is to consider the tensor product $K_N := \otimes_{n=1}^N K$ on $\otimes_{n=1}^N \mathcal{H}$. Since for $h_1, \ldots, h_N \in \mathbb{N}$ the product $\lambda_{h_1}(K) \cdots \lambda_{h_N}(K)$ is an eigenvalue of $K_N$ we obtain from (6) that

$$\begin{aligned} \left( \sum_n |\lambda_n(K)|^{1+\varepsilon} \right)^N &= \sum_{h_1,\ldots,h_N} |\lambda_{h_1}(K) \cdots \lambda_{h_N}(K)|^{1+\varepsilon} \\ &\leq \sum_n |\lambda_n(K_N)|^{1+\varepsilon} \leq C(\varepsilon)\|K_N\|_{\mathrm{tr}} = C(\varepsilon)\|K\|_{\mathrm{tr}}^N. \end{aligned}$$

Taking the $N$th root on both sides and sending first $N \to \infty$ and then $\varepsilon \to 0$ we obtain estimate (5).

Now let us consider a more general setting: Suppose that $A$ is a selfadjoint bounded operator on $\mathcal{H}$ and $K$ is trace class (and not selfadjoint). Note that in this case the non-real spectrum of $B = A + K$ consists of discrete eigenvalues only. Further suppose that for every $\varepsilon > 0$ we have an estimate of the form

$$\sum_{\lambda \in \sigma_d(B) \cap \sigma(A)^c} \mathrm{dist}(\lambda, \sigma(A))^{1+\varepsilon} \leq C(\varepsilon)\|K\|_{\mathrm{tr}}.$$

**Q1:** Does this imply that

$$\sum_{\lambda \in \sigma_d(B) \cap \sigma(A)^c} \mathrm{dist}(\lambda, \sigma(A)) \leq \|K\|_{\mathrm{tr}} \ ?$$

A positive answer to this question would lead to improvements of existing Lieb-Thirring type inequalities for non-selfadjoint Schrödinger- and Jacobi-Operators.

# 13 M. Hitrik

I am mainly interested in the spectral theory of non-selfadjoint differential operators, especially in the semiclassical limit. The following are some issues related to the subject of the workshop, that I would be very happy to see discussed at the workshop.

## 13.1 Upper bounds on the norm of the resolvent

It is well known that the spectrum of a non-selfadjoint operator does not control its resolvent and that the latter may become very large far from the spectrum. Some general upper bounds on resolvents are provided by the abstract operator theory, and restricting the attention to the setting of semiclassical operators on $\mathbf{R}^n$, let us give a rough statement of such bounds. Assume that $P = p^w(x, hD_x)$ is the semiclassical Weyl quantization on $\mathbf{R}^n$ of a nice symbol $p$ with $\mathrm{Re}\, p \geq 0$, say. Then the norm of the resolvent of $P$ is bounded from above by a quantity of the form $\mathcal{O}(1)\exp(\mathcal{O}(1)h^{-n})$, provided that $z \in \mathrm{neigh}(0, \mathbf{C})$ is not too close to the spectrum of $P$. On the other hand, the available lower bounds on the resolvent of $P$, in the interior of the range of the symbol, coming from the pseudospectral considerations, are typically of the form $C_N^{-1}h^{-N}$, $N \in \mathbf{N}$, or $(1/C)e^{1/(Ch)}$, provided that $p$ enjoys some analyticity properties, [DSZ04]. There appears to be therefore a substantial gap between the available upper and lower bounds on the resolvent, especially when $n \geq 2$, which, to the best of my knowledge, has so far only been bridged in the very special case of elliptic quadratic differential operators, see [HSV13].

### 13.2 Inverse spectral problems for non-selfadjoint operators, especially in the semiclassical limit

Given a suitable $h$-pseudodifferential operator $P = p^w(x, hD_x)$ on $\mathbf{R}^n$ or a compact manifold, we would like to understand what information about the classical symbol $p$ can be determined from the spectrum of $P$, in the semiclassical limit $h \to 0$. We are especially interested in cases when $P$ is non-selfadjoint, with the inverse problems for resonances and for damped wave equations being important sources of motivation. See [DH12], [Hal13], [Pha] for some of the recent works on semiclassical inverse spectral problems in the non-selfadjoint setting.

### 13.3 Spectra for non-selfadjoint operators in the presence of symmetries

The proof of the reality of the exponentially small eigenvalues of the Kramers-Fokker-Planck type operators in [HHS11] depends on a reflection symmetry for such operators, and there are many natural non-selfadjoint situations where symmetries play a role, including PT-symmetric operators and operators with supersymmetric structures. See also [Shi02], [KS02].

## 14 D. Krejčiřík

### 14.1 Large-time behaviour of the heat equation: subcriticality versus criticality

This open problem is a repetition of the open problem raised during previous meetings in Prague (2010) and Barcelona (2012)

`www.ujf.cas.cz/ESFxNSA/`
`http://gemma.ujf.cas.cz/~david/OTAMP2012/OTAMP2012.html`

but little progress has been made so far. Please visit the links above for more details and references.

Our conjecture is that the solutions of the heat equation "decay faster" for large times provided that the generator is "more positive" in the sense of the validity of a Hardy-type inequality. There exist both semigroup (with Zuazua, 2010) and heat-kernel (with Fraas and Pinchover, 2010) versions of the conjecture and the latter involves non-self-adjoint operators too. The conjecture has been supported by several particular situations, but there exist no general result yet.

### 14.2 The cloaking effect in metamaterials: beyond ellipticity

In mathematical models of metamaterials characterised by negative electric permittivity and/or negative magnetic permeability, there appear operators of the type "div sgn grad".

**Q1:** How to define such an operator as a self-adjoint operator in an $L^2$ setting?

There exist numerous works on a changed problem in which there is a small complex constant added to the minus one in the sign function. The only exception (apart from the one-dimensional situation, which is elementary) seems to be my recent joint paper with Behrndt [BK14].

Here we solve the original problem for a particular geometry (rectangle) with help of a refined extension theory. It turns out that the domain of the self-adjoint operator is not a subset of the Sobolev space $H^1$ and there is an essential spectrum (although the geometry is bounded).

### 14.3 Semiclassical pseudomodes of Schrödinger operators with discontinuous potentials

For smooth potentials there exists a quite general theory on the construction of semiclassical pseudomodes, see [DSZ04].

**Q1:** Can the technique be adapted to discontinuous potentials?

In my joint paper with Henry [HK15] we have a non-trivial pseudospectrum in a toy model (complex Heaviside-type potential). However, our technique is restricted to the particular situation and the non-trivial pseudospectrum is rather generated by the behaviour of the potential at infinity.

## 15 M. Levetin

My principal interests related to the workshop are in spectral problems for linear pencils $A - \lambda B$ where both coefficients $A$, $B$ are self-adjoint sign-indefinite operators. In this case, the pencil spectra are generally speaking non-real, and localisation of eigenvalues and their asymptotic behaviour (with respect to some parameters which may be present in the problem) lead to difficult and interesting problems combining operator theory, complex analysis, special functions and many other topics.

I will pose some open problems arising in two realisations of the pencil - a matrix operator first studied by Davies and Levitin (2014), with interesting challenges in its random analogues, and an indefinite Sturm-Liouville pencil by Behrndt et al. for which partial results were obtained in Levitin-Seri (2015). Despite apparent simplicity of these models, they turn out to be remarkably deep and difficult.

## 16 M. Marletta

Finite section and other approximation methods are often used to approximate the spectra of operators. In the case of self-adjoint operators, relatively mild conditions ensure that the whole spectrum can be approximated, though usually spectral pollution is a problem.

In the non-selfadjoint case, E.B. Davies pointed out that applying a finite section method to the differential operator $-d^2/dx^2 + d/dx$ in $L_2(\mathbb{R})$, truncating it to $L_2(-X, X)$ with Dirichlet conditions at $\pm X$, will give only positive real eigenvalues, whereas the true spectrum is a parabola. It is therefore interesting to examine for what classes of non-selfadjoint operators finite section methods always manage to capture all of the essential spectrum. Special cases examined so far include pseudo-ergodic Jacobi matrices; and both trace-class and relative trace-class perturbations of self-adjoint Jacobi matrices and one-dimensional Schrödinger operators.

## 17 M. Shubov

In recent years my research was focused on the following main topics.

### 17.1 Non-selfadjoint operators that are the dynamics generators for various models of elastic structures

The list of these models include:

(i) equations of damped string with non-constant coefficients;

(ii) Euler-Bernoulli, Rayleigh, and Timoshenko beam models with dissipative boundary conditions;

(iii) bending-torsion vibration model that is used in aero-elasticity to model ground vibrations of a long slender wing;

(iv) mathematical models of multi-walled carbon nano-tubes;

(v) mathematical models of energy harvesters using the idea of transforming mechanical energy of structural vibration into electrical circuit energy.

Each of the above models can be represented as a linear evolution equation in a Hilbert space. This space is the state space of the corresponding system equipped with the energy metric. The results include:

(i) asymptotics of the spectrum;

(ii) asymptotics of the generalized eigenvectors;

(iii) proof of the Riesz basis property of the generalized eigenvectors.

The Riesz basis property analysis usually includes three steps: (i) proof of minimality; (ii) proof of the fact that the generalized eigenvectors form a Riesz basis in their closed linear span; (iii) proof of the completeness of the generalized eigenvectors in the state space; step (iii) is usually most complicated. Here various approaches are possible depending on a specific system, e.g., using the Krein's theorem about dissipative operators with nuclear imaginary part or using estimates of the resolvent oerator.

### 17.2 Physically realistic model of an aircraft wing

A series of my works is devoted to a quite complicated physically realistic model of an aircraft wing in a surrounding subsonic airflow. This model is represented as an evolution-convolution equation in a Hilbert state space. The spectral analysis of such equation can be reduced to a study of the so-called generalized resolvent operator, which is not a resolvent of any operator but is a finite-meromorphic operator-valued function of the spectral parameter. The poles of this function correspond to the aero-elastic vibration modes and residues at these poles are defined in terms of the corresponding mode shapes. The main results are devoted to:

(i) asymptotics of the aero-elastic modes;

(ii) Riesz basis property of the mode shapes;

(iii) existence and location of the unstable (fluttering) modes.

### 17.3 Controlability problems

Another direction of my work deals with various exact and approximate controlability problems for all the aforementioned systems. It is related to a specific area of harmonic analysis dealing with geometric properties (minimality, completeness, and unconditional basis property) of non-harmonic exponentials.

### 17.4 Resonances in quantum and acoustical scattering

In the past I also worked on resonances in quantum and acoustical scattering. In particular I derived asymptotic formulas for resonances of 3-dim Schrödinger operator with non-spherically symmetric potential. I have also studied asymptotics of resonances and geometry of resonance states for the problem of a non-homogeneous damped string with non-constant density and damping coefficients. The results, in particular, show that when the density coefficient has a discontinuity, the set of the resonance states forms an unconditional basis. In some cases the aforementioned bases are close to orthonormal bases of the state space. These bases are the generalization of the Bari bases.

I have a number of open questions for each of the above mentioned research topics.

## 18 P. Siegl

My interests in non-self-adjoint (mainly differential) operators include

(i) spectra and pseudospectra and their approximations, presence of symmetries,

(ii) basis properties of eigensystem (perturbation results like below, growing spectral projection norms),

(iii) Laplacians on graphs with non-self-adjoint boundary conditions,

(iv) damped wave equation with unbounded damping.

## 18.1 Perturbations of harmonic oscillator

There are several recent works on the basis properties of perturbations of harmonic oscillator type operators, i.e. $A = A^* \geq 0$, the spectrum of $A$ is discrete, all eigenvalues $\{\mu_n\}_n$ are simple and $\mu_{n+1} - \mu_n \geq \delta > 0$, see [AM12a, Shk10, AM12b, MS13] and related works on asymptotics of eigenvalues [Shk12, Mit15]. The main assumption on the (unbounded in general) perturbation $B$ guaranteeing that the eigensystem of $A + B$ contains a Riesz basis is the so-called local subordination condition:

$$\lim_{n\to\infty} \|B\psi_n\| = 0 \qquad \text{or} \qquad |\langle B\psi_m, \psi_n\rangle| \leq \frac{C}{m^\alpha n^\alpha} \quad \text{with some } C > 0,\ \alpha > 0, \tag{7}$$

where $\{\psi_n\}_n$ are normalized eigenvectors of $A$ related to eigenvalues $\{\mu_n\}_n$.

When applied to potential perturbations of the harmonic oscillator $A_0$ in $L^2(\mathbb{R})$,

$$A_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2, \qquad \mathrm{Dom}(A) = \{\psi \in W^{2,2}(\mathbb{R}) \,:\, x^2 f \in L^2(\mathbb{R})\}, \tag{8}$$

it can be concluded that the eigensystem of $A + V$ contains a Riesz basis if $V \in L^p(\mathbb{R})$ with $1 \leq p < \infty$, (the second condition in (7) is satisfied also for some distributional potentials as $\delta(x)$), see [AM12a, MS13].

Nonetheless, it remains open what happens for $V \in L^\infty(\mathbb{R})$ (which is obviously a bounded perturbation unlike other $L^p$ potentials in general). It is known that

(i) the eigensystem of $A_0 + V$, where $V$ is a bounded operator (not necessarily a potential), contains a Riesz basis of finite dimensional subspaces (the estimate on the dimension of subspaces is not given), see [Agr94],

(ii) there exists a bounded operator $V$ (*not* a potential) with $\|V\| = 1$ (hence the spectrum of $A_0 + V$ is real) such that the eigensystem of $A_0 + V$ does not contain even a basis, see [AM12a],

(iii) under symmetry assumptions (like PT-symmetry or other symmetry in a Krein space) on a bounded operator $V$ (not necessarily a potential), the eigensystem of $A_0 + V$ contains a Riesz basis if $\|V\| < \frac{2}{\pi}$, see [AMS09, AMT10].

**Q1:** What can be said about the eigensystem of $A_0 + V$, $V \in L^\infty(\mathbb{R})$, without the symmetry and the small norm assumptions? Is there a *potential* $V \in L^\infty(\mathbb{R})$ such that the eigensystem of $A_0 + V$ does not contain a Riesz basis or even a basis?

# 19 D. A. Smith

## 19.1 Overview

My primary interest is non-self-adjoint differential operators where the non-self-adjointness arises as a result of boundary conditions. I became interested in this topic through the study of Fokas' Unified Transform Method for initial-boundary value problems for constant-coefficient linear evolution equations in 1-space, 1-time domains. Typical equations are the heat, linear Schrödinger, and linearized KdV equations. The method has the remarkable property that it can be used to solve any such well-posed problem, but the resulting solution representation was poorly understood.

## 19.2 Spectral representation of two-point differential operators

Following Birkhoff's [Bir08b, Bir08a] classification of two-point differential operators into regular and irregular, [Loc08] refined the classification of irregular operators to simply irregular and degenerate irregular. The eigenfunction basis properties of regular and simply irregular operators are well understood. However, for a certain degenerate irregular operator,

$$S : \{f \in C^\infty[0,1] : f(0) = f(1) = f'(1) = 0\} \to C^\infty[0,1], \qquad Sf = f''',$$

Jackson (1915) showed that eigenfunction expansions diverge.

It is now understood [SF13] that divergence of such eigenfunction expansions does not preclude the spectral representation of degenerate irregular operators. Rather, instead of using traditional eigenfunctions, one must look to a new species of spectral functional: augmented eigenfunctions.

**Definition 2.** Let $I$ be an open real interval and let $C$ be a linear topological space of functions defined on the closure of $I$ with sufficient smoothness and decay conditions. Let $\Phi \subseteq C$ and let $L : \Phi \to C$ be a linear operator. Let $\gamma$ be an oriented contour in $\mathbb{C}$ and let $E = \{E_\lambda : \lambda \in \gamma\}$ be a family of functionals $E_\lambda \in C'$. Suppose there exist corresponding *remainder* functionals $R_\lambda \in \Phi'$ and *eigenvalues* $\lambda^n$ such that

$$E_\lambda(L\phi) = \lambda^n E_\lambda(\phi) + R_\lambda(\phi), \qquad \forall \phi \in \Phi, \forall \lambda \in \gamma. \tag{9}$$

If

$$\int_\gamma e^{i\lambda x} R_\lambda(\phi)\, \mathrm{d}\lambda = 0, \qquad \forall \phi \in \Phi, \forall x \in I, \tag{10}$$

then we say $E$ is a family of *type* I *augmented eigenfunctions* of $L$ up to integration along $\gamma$.

If

$$\int_\gamma \frac{e^{i\lambda x}}{\lambda^n} R_\lambda(\phi)\, \mathrm{d}\lambda = 0, \qquad \forall \phi \in \Phi, \forall x \in I, \tag{11}$$

then we say $E$ is a family of *type* II *augmented eigenfunctions* of $L$ up to integration along $\gamma$.

In the theory of pseudospectra it is required that the norm of the remainder functional $R_\lambda(\phi)$ is small, whereas in our definition it is required that the integral of $\exp(i\lambda x)R_\lambda(\phi)$ along the contour $\gamma$ vanishes.

It has been shown that, in the following senses, augmented eigenfunctions provide spectral representations of certain degenerate irregular operators, including the operator studied by Jackson.

**Definition 3.** Suppose that $E = \{E_\lambda : \lambda \in \gamma\}$ is a system of type I augmented eigenfunctions of $L$ up to integration over $\gamma$, and that

$$\int_\gamma e^{i\lambda x} E_\lambda L\phi\, \mathrm{d}\lambda \text{ converges } \forall \phi \in \Phi, \forall x \in I. \tag{12}$$

Furthermore, assume that $E$ is a complete system. Then we say that $E$ provides a *spectral representation* of $L$ in the sense that

$$\int_\gamma e^{i\lambda x} E_\lambda L\phi\, \mathrm{d}\lambda = \int_\gamma e^{i\lambda x} \lambda^n E_\lambda \phi\, \mathrm{d}\lambda \qquad \forall \phi \in \Phi, \forall x \in I. \tag{13}$$

**Definition 4.** Suppose that $E^{(\mathrm{I})} = \{E_\lambda : \lambda \in \gamma^{(\mathrm{I})}\}$ is a system of type I augmented eigenfunctions of $L$ up to integration over $\gamma^{(\mathrm{I})}$ and that

$$\int_{\gamma^{(\mathrm{I})}} e^{i\lambda x} E_\lambda L\phi\, \mathrm{d}\lambda \text{ converges } \forall \phi \in \Phi, \forall x \in I. \tag{14}$$

Suppose also that $E^{(\mathrm{II})} = \{E_\lambda : \lambda \in \gamma^{(\mathrm{II})}\}$ is a system of type II augmented eigenfunctions of $L$ up to integration over $\gamma^{(\mathrm{II})}$ and that

$$\int_{\gamma^{(\mathrm{II})}} e^{i\lambda x} E_\lambda \phi\, \mathrm{d}\lambda \text{ converges } \forall \phi \in \Phi, \forall x \in I. \tag{15}$$

Furthermore, assume that $E = E^{(\mathrm{I})} \cup E^{(\mathrm{II})}$ is a complete system. Then we say that $E$ provides a *spectral representation* of $L$ in the sense that

$$\int_{\gamma^{(\mathrm{I})}} e^{i\lambda x} E_\lambda L\phi\, \mathrm{d}\lambda = \int_{\gamma^{(\mathrm{I})}} \lambda^n e^{i\lambda x} E_\lambda \phi\, \mathrm{d}\lambda \qquad \forall \phi \in \Phi, \forall x \in I, \tag{16a}$$

$$\int_{\gamma^{(\mathrm{II})}} \frac{1}{\lambda^n} e^{i\lambda x} E_\lambda L\phi\, \mathrm{d}\lambda = \int_{\gamma^{(\mathrm{II})}} e^{i\lambda x} E_\lambda \phi\, \mathrm{d}\lambda \qquad \forall \phi \in \Phi, \forall x \in I. \tag{16b}$$

According to Definition 3, the operator $L$ is diagonalized (in the traditional sense) by the complete transform-inverse transform pair

$$\left( E_\lambda, \int_\gamma e^{i\lambda x} \cdot \, \mathrm{d}\lambda \right). \tag{17}$$

Hence, augmented eigenfunctions of type I provide a natural extension of the generalized eigenfunctions employed by Gel'fand. However, augmented eigenfunctions of type II are clearly quite different from generalized eigenfunctions. Definition 4 describes how an operator may be decomposed into two parts, one of which is diagonalized in the traditional sense, whereas the other possesses a diagonalized inverse.

## 19.3 Augmented eigenfunctions in Fokas' method

When one applies separation of variables to solve the Dirichlet problem for the heat equation on a finite interval, one implicitly uses the fact that expansion in the eigenfunctions of the spatial (ordinary) differential operator are convergent. Consider the third order analogue:

$$\begin{aligned}
[\partial_t + \partial_x^3]q(x,t) &= 0, & (x,t) &\in (0,1)\times(0,T),\\
q(x,0) &= q_0(x), & x &\in [0,1],\\
q(0,t) &= 0, & t &\in [0,T],\\
q(1,t) &= 0, & t &\in [0,T],\\
\partial_x q(1,t) &= 0, & t &\in [0,T],
\end{aligned}$$

where the initial datum $q_0 \in C^\infty[0,1]$ is compatible with the boundary conditions. As the spatial part of this initial-boundary value problem is Jackson's operator $S$, it is no surprise that separation of variables will not yield a solution to this problem.

Nevertheless, this problem can be solved using the Unified Transform Method of Fokas (for a recent pedagogical introduction, see [DTV14]; for a bibliography and list of applications, see `http://unifiedmethod.azurewebsites.net`). The solution is represented using an augmented eigenfunction expansion of the initial datum, with simple time dependence:

$$q(x,t) = \int_\Gamma e^{i\lambda x + i\lambda^3 t} E_\lambda(q_0)\,\mathrm{d}\lambda, \tag{18}$$

where $\Gamma$ is a certain complex contour, and $E_\lambda$ are a complete family of augmented eigenfunctions of Jackson's $S$, indexed by $\Gamma$. In this way, we see that the augmented eigenfunctions describe a forward transform, with $\int_\Gamma e^{i\lambda x}\cdot\,\mathrm{d}\lambda$ the corresponding inverse transform. Augmented eigenfunctions were first defined [SF13] as a way to describe the spectral meaning of this solution representation. Similar results were obtained for problems with half-line spatial domain [PS14]. The finite interval and half-line results were compared and contrasted in [Smi14].

In the context of initial-boundary value problems, the meaning of the control on the remainder functionals becomes clear. The generalized eigenfunctions of the half-line Dirichlet heat operator encode the Fourier sine transform. This transform is useful for solving the initial-boundary value problem because, as a consequence of the transform diagonalizing the operator, the time evolution in spectral space is very simple. With augmented eigenfunctions, the operator is not diagonalized as a nonzero remainder functional is permitted, so the time evolution in spectral space is not simple. However, the control equation (10) or (11) ensures that the nonsimple part of the time evolution vanishes upon application of the inverse transform.

Crucially, the completeness results on augmented eigenfunctions, which are necessary to claim a valid spectral representation of an operator, are obtained through Fokas' method.

## 19.4 Open problems

The current augmented eigenfunction results are for two-point differential operators of the form

$$S : \{f \in C^\infty[0,1] : B_j\phi = 0 \forall j \in \{1,2,\dots,n\}\} \to C^\infty[0,1], \qquad S\phi(x) = (-i)^n \frac{\mathrm{d}^n\phi}{\mathrm{d}x^n}(x) \tag{19}$$

where $n \geq 2$ is an integer, and $B_j$ are linearly independent two-point boundary forms, and also for the corresponding one-point differential operators.

Fokas' method has been formulated for constant-coefficient operators which are not equal to their principal parts [Pel04], and operators with rational symbol [FP05, VD11]). It is expected that the only technical hurdle in extending the augmented eigenfunction results to such settings will be in permitting the operator $S$ to be formally non-self-adjoint.

There has not been a satisfactory implementation of Fokas' method for variable-coefficient operators, or operators on less smooth spaces, however eigenfunction expansion results for regular and simply irregular operators are routinely formulated in such settings. It is therefore desirable to find a new way to prove completeness results for augmented eigenfunctions that may be extensible to these wider classes of differential operators. Such an advance may also open the way to studying initial-boundary value problems for equations with variable coefficients.

The definition of augmented eigenfunctions need not be restricted to differential operators, but could conceivably be profitably applied to other non-self-adjoint linear operators whose eigenfunctions do not provide convergent expansions.

# 20 R. Weikard

I am interested in inverse scattering theory for the Schrödinger equation, in particular in the inverse resonance problem. While the free problem is self-adjoint the relevant methods allow for non-self-adjoint perturbations. Inverse resonance problems take as input the location of eigenvalues and resonances, data which — in a practical sense — are more readily available then reflection coefficients or scattering phases. The methods also allow to establish stability theorems allowing for finite noisy data.

Open problems in this context exist for inverse scattering problems on quantum trees or, more generally, Schrödinger equations with matrix- or operator-valued coefficients.

# 21 Open problems since 2010

Please see open problems at the webpages of previous workshops (Prague, Edinburgh, Barcelona).

`http://www.ujf.cas.cz/ESFxNSA/`
`http://www.ujf.cas.cz/MAPwNSA/`
`http://gemma.ujf.cas.cz/~david/OTAMP2012/OTAMP2012.html`

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