2011 Annual Report on the Second ARCC Grant

Submitted to the National Science Foundation in partial fulfillment of the reporting requirements for grant DMS-0637285

Contents

1. Overview

This is the annual report from the second year of the second five-year ARCC grant. This reporting period includes budgeting for 18 workshops and 8 SQuaREs.

A primary goal of the current ARCC grant is to develop and refine the SQuaREs program. SQuaREs, described in more detail in Section 3, foster new collaborations on ambitious research projects over a multi-year period. During the past year, eight SQuaREs with a total of 49 participants met at AIM. Feedback about the program has so far been extremely positive, and the AIM Directors have also made significant progress in understanding the dynamics of this program to ensure its continued success.

Beginning to disseminate the AIM style of workshop is another goal of the current grant. This year, we took another step toward this goal by co-hosting a workshop ("Volume entropy rigidity," Workshop 16 in this report) at ETH in Zurich. This workshop, organized by Marc Burger, Francois Ledrappier, Seonhee Lim, and Gabriele Link, was mathematically successful and received very positive evaluations from participants. AIM Directors Brian Conrey and Estelle Basor attended the workshop and worked with the organizers throughout the week to facilitate the AIM format, for example by assisting the organizers to structure events and discussions in a way that led to the formation of productive working groups.

This year, we continued our commitment to initiatives intended to increase participation in and diversity of the mathematical workforce. Some of these initiatives, including a program called Research Experiences for Undergraduate Faculty (REUF), are described in detail in Sections 5 and 6, which also describe AIM's ongoing efforts to include a diverse group of participants in all of our programming. The Math Teachers' Circle program, intended to increase middle school teachers' knowledge of mathematical problem solving, expanded through two "How to Run a Math Teachers' Circle" workshops in Summer 2009, one funded through NSA and the other as an extra outreach workshop in addition to the year's 17 scientific workshops. Of the 13 teams that attended, at least 12 have plans to begin their own Math Teachers' Circle by Summer 2010. In addition to these ongoing workforce initiatives, we also continued developing tools to strengthen the research infrastructure, supported by the CDI grant "Bibliographic Knowledge Network." Particularly relevant to ARCC was the ongoing development of the Problem List tool, which has the potential to be a great resource for our workshops, and, we believe, will further stimulate the organization of mathematical problems.

In the future, we plan to continue refining the SQuaREs program and also intend to hold one or two workshops per year in a remote location for dissemination purposes. We will continue our work on infrastructural tools that will improve our workshops and will also work on initiatives that will broaden participation in the mathematical sciences workforce. In addition, we will continue our involvement with new initiatives that might benefit from the AIM style of workshop.

2. Workshops and SQuaRES in this report

Workshops:

1. Noncongruence modular forms and modularity
August 17 to August 21, 2009
Winnie Li, Tong Liu, Ling Long, and Ravi Ramakrishna

2. Relative trace formula and periods of automorphic forms
August 24 to August 28, 2009
Herve Jacquet, Erez Lapid, and Akshay Venkatesh

3. Permanents and modeling probability distributions
August 31 to September 4, 2009
Alon Orlitsky, Narayana Santhanam, and Krishnamurthy Viswanathan

4. Convex algebraic geometry, optimization and applications
September 21 to September 25, 2009
William Helton and Jiawang Nie

5. Rational curves and A1-homotopy theory
October 5 to October 9, 2009
Aravind Asok and Jason Starr

6. Higher Reidemeister Torsion
October 19 to October 23, 2009
Sebastian Goette, Kiyoshi Igusa, and John Klein

7. Implementing algebraic geometry algorithms
October 26 to October 30, 2009
Hirotachi Abo, Anton Leykin, Sam Payne, and Amelia Taylor

8. The Cuntz semigroup
November 2 to November 6, 2009
Nate Brown, George Elliott, and Andrew Toms

9. Cyclic homology and symplectic topology
November 9 to November 13, 2009
Mohammed Abouzaid, Eleny Ionel, Lenhard Ng, and Paul Seidel

10. Brownian motion and random matrices
December 14 to December 18, 2009
Peter Forrester, Brian Rider, and Balint Virag

11. Mock modular forms in combinatorics and arithmetic geometry
March 8 to March 12, 2010
Kathrin Bringmann, Ken Ono, and Sander Zwegers

12. Localization techniques in equivariant cohomology
March 15 to March 19, 2010
William Fulton, Rebecca Goldin, and Julianna Tymoczko

13. Computational optimization for tensor decompositions
March 29 to April 2, 2010
Rasmus Bro, Michael Friedlander, Tamara G. Kolda, and Stephen Wright

14. Supercharacters and combinatorial Hopf algebras
May 17 to May 21, 2010
Nantel Bergeron, Persi Diaconis, Jean-Yves Thibon, and Nathaniel Thiem

15. Low dimensional structures in dynamical systems with variable time lags
June 7 to June 11, 2010
John Mallet-Paret, Roger Nussbaum, and Hans-Otto Walther

16. Control and optimization of open quantum systems for information processing
June 21 to June 25, 2010
Constantin Brif, Matthew Grace, Andrew Landahl, and Wayne Witzel

17. How to run a Math Teachers' Circle (AIM)
June 28 to July 2, 2010
Tatiana Shubin, Tom Davis, and Joshua Zucker

18. Components of Hilbert Schemes
July 19 to July 23, 2010
Robin Hartshorne, Diane Maclagan, and Gregory G. Smith

19. Differentiable structures on finite sets
August 2 to August 6, 2010
Charles Fefferman and Nahum Zobin

SQuaREs:

1. Algebraic topology and physics
June 1 to June 5, 2009
Hisham Sati

2. Augmentations, rulings, and generating families
June 1 to June 5, 2009
Lisa Traynor

3. The Baum-Connes conjecture and geometric representation theory
July 13 to July 17, 2009
Jonathan Block

4. Ehrhart functions
July 13 to July 17, 2009
Jesus De Loera

5. Arrangements, logarithmic vector fields, and D-modules
February 8 to February 12, 2010
Uli Walther

6. Hybrid optimization methods
March 1 to March 5, 2010
Katie Fowler

7. Isometric embedding of higher dimensional Riemannian manifolds
March 8 to March 12, 2010
Marshall Slemrod

8. Algebraic K-groups
March 22 to March 26, 2010
Michael Hill

9. Higher divergence functions
April 26 to April 30, 2010
Moon Duchin

10. Directional discrepancy
May 3 to May 7, 2010
Dmitriy Bilyk

11. Geometry and physics of the new topological recursion
May 3 to May 7, 2010
Motohico Mulase

12. The Kadison-Singer problem
May 10 to May 14, 2010
Pete Casazza

13. Modeling cortical spreading depression
May 24 to May 28, 2010
Robert Miura

14. Towards solving Smale's sixth problem
May 24 to May 28, 2010
Marshall Hampton

15. Hausdorff geometry of complex polynomials, positive charge distributions and normal operators
July 12 to July 16, 2010
Mihai Putinar

16. Nonsmooth and nonconvex optimization problems with matrix variables
July 12 to July 16, 2010
Jane Ye

3. The SQuaREs program

AIM initiated a new program called SQuaREs (Structured Quartet Research Ensembles) in the fall of 2007. The purpose of SQuaREs is to allow a small group of mathematicians to collaborate for a week at AIM on an ambitious research project with the opportunity for returning in subsequent years. The AIM staff works closely with the organizers and has adapted many of the techniques the AIM workshop program to help the participants map out strategies, set priorities and set in place a framework for progress on significant research.

In year three of the current ARCC grant, AIM hosted 14 different SQuaRE groups with a total of 68 participants. This included seven first week SQuaREs, six second week meetings and one third week meeting. The SQuaRE groups typically bring a mixture of senior and junior mathematicians together, some who have never worked together before and others who have begun some initial collaborations at AIM workshops. The program allows for week long investigations without interruptions. and allows new unproven collaborations to flourish.

Here are some observations about three SQuaREs that illustrate the richness of topics and the uniqueness and the success of the program. These SQuaREs took place during year three of the ARCC grant. One involves a very pure area of mathematics, one a very applied area, and one is a blend of pure mathematics and physics.

This AIM SQuaRE "Modeling Cortical Spreading Depression" focused on the mathematical modeling of cortical spreading depression (CSD), a slow wave phenomenon in the cortex of different brain structures. While there are many interesting aspects of CSD, its connection to migraine (with aura) in humans makes it a particularly important neurophysiological phenomenon from the clinical point of view. The pain-generating mechanism of migraines is not well-understood, and it has been hypothesized that cerebral vasoconstriction following CSD may be partially responsible. The SQuaRE objective was to derive a mathematical model for CSD in which the effects of cerebral blood flow on the CSD activity are taken into account.

This SQuaRE brought together an interesting mix of mathematical modelers, a research neurologists, and numerical analysts. It met for one week in 2010 and made significant progress in forming a mathematical model for CSD which includes the mathematical components of membrane ionic currents, water movement, and the vascular system. The mathematical and computational analysis of this system of equations will be explored in future meetings.

The SQuaRE, "Geometry and physics of the new topological recursion," held its first week in May of 2010 illustrates how AIM is equipped to respond quickly to fast moving mathematical developments. This SQuaRE grew out of a workshop, "A New Recursion Structure in Topological String Theory and Enumerative Geometry," held in June of 2009. The workshop was aimed at solving some of the concrete conjectures in Gromov-Witten theory, one known as the BKMP conjecture (due to Bouchard, Klemm, Marino, and Pasquetti) and in the simplest case of the BKMP known as the Bouchard-Marino conjecture. The goal of the workshop was to combine the recursion ideas of random matrix theory and algebraic geometry. At this SQuaRE meeting, one of the participants reported his refined theory of topological recursion. Another participant presented his solution to the simplest care of the BKMP conjecture. The group then identified what would be necessary to carry the known methods to the general case and what was a common mathematical structure of the solved cases. Much progress was made in these directions.

Here is a remark from the organizer report:

"The SQuaRE really functioned well and served an unforeseen purpose: to make researchers of totally different background to be able to scientifically communicate and to formulate possible theorems. Without the AIM meeting the language barrier would have prevented the two sides (physics and mathematics) to communicate.

The SQuaRE "Algebraic K-Groups" held its first meeting at AIM in March of 2010. The participants created a list of five interesting problems to consider at the beginning of the SQuaRE. For example, one of these was to provide a conceptual description of cyclotomic (equivariant) structure on the topological Hochschild homology of Thom spectra, and use this to study the topological cyclic homology of Thom spectra. The problems were highly interrelated, and according to the report "much of our time at the SQuaRE was spent teasing apart the connections and dependencies". By the end of the week, the above problem was solved and three of the others had partial solutions and additional conjectures were formulated. This SQuaRE will return in June of 2011.

4. Selection of new Workshops and SQuaREs

The annual AIM Scientific Board meeting was held December 6, 2008 and was chaired by Charles Fefferman of Princeton University. Board members present were

- Mladen Bestvina
- Robert Calderbank
- Charles Fefferman
- David Gabai
- Alexander Kechris
- Sandor Kovacs
- Joyce R. McLaughlin
- David Siegmund
- Jean Taylor
- Pauline van den Driessche
- Dan-Virgil Voiculescu

Written reports were received from:

- Gerard Ben Arous
- Robbert Dijkgraaf
- Yakov Eliashberg
- Fan Graham
- Victor Guillemin
- Yuval Peres
- Paul Rabinowitz
- Peter Sarnak
- Efim Zelmanov

Present from the AIM staff were:

- Estelle Basor
- Brian Conrey
- David Farmer
- Leslie Hogben

The first part of the meeting was devoted to workshop selection. After discussion, the workshops chosen for funding were:

1 Supercharacters and combinatorial Hopf algebras
2 A New Recursion Structure in Topological String Theory and Enumerative Geometry
3 Low Dimensional Structures in Dynamical Systems with Variable Time Lags
4 Computational Optimization for Tensor Decompositions
5 Cyclic homology and symplectic topology
6 Rational curves and A1-homotopy theory
7 Convex Algebraic Geometry, Optimization and Applications
8 Hurwitz spaces, covers, and mapping class groups: algebraic geometry, topology, Arithmetic
9 Localization techniques in equivariant cohomology
10 Mock modular forms in combinatorics and arithmetic geometry

11 Computation of permanents and modeling probability distributions
12 Statistics and computational topology
13 Geodesics
14 Implementing Algorithms in Macaulay
15 Brownian Motion and Random Matrices
16 The Cuntz Semigroup

The second part was devoted to the selection of SQuaREs. The SQuaREs chosen were:

1 Ehrhart functions: at the heart of the intersection of Algebra, Combinatorics, and Complexity Theory (organizer Jesus de Loera)
2 Higher divergence functions (organizer Moon Duchin)
3 Hybrid optimization methods for simulation-based problems (organizer Kathleen Fowler)
4 Towards Solving Smale's Sixth Problem (organizer Marshall Hampton)
5 Modeling cortical spreading depression (organizer Robert Muira)
6 Isometric embeddings of higher dimensional Riemannian manifold (organizer Marshall Slemrod)
7 Augmentations, Rulings, and Generating Families (organizer Lisa Traynor)
8 Arrangements, logarithmic vector fields, and D-modules, (organizer Uli Walther)

Because of the large number of competitive SQuaREs, it was decided to increase the number of SQuaREs and have only two possible spots reserved for hot topic workshops.

5. Diversity efforts

The accessibility of AIM resources to the entire mathematical community is an integral part of AIM's mission. Our efforts to ensure inclusion of traditionally under-represented groups such as women, racial/ethnic minorities, faculty at primarily undergraduate institutions, and junior researchers, have two foci.

(1) Ensuring full inclusion in AIM workshops and SQuaREs.
(2) Undertaking new initiatives that support inclusion of diverse groups of mathematical researchers and enhance the mathematical workforce.

5.1. **Full inclusion in AIM workshops and SQuaREs.** AIM has for many years been successful in ensuring inclusion of diverse groups of researchers in ARCC workshops. Key ingredients of this success are

- Recruiting of diverse groups of organizers by AIM Directors and members of the Human Resources (HR) Board
- Consideration of diversity issues by the Scientific Board when selecting workshops through input from the HR Board.
- The tools created by AIM to assist in finding diverse groups of researchers who might otherwise be overlooked.
- The involvement of AIM Directors in all stages of workshop development.
- An open application process for one quarter of the places.

The HR Board is charged with increasing the participation of diverse groups in the activities of ARCC, by identifying/encouraging diverse groups of potential organizers for workshops/programs, providing input on the ARCC workshop proposals to the Scientific Board, and suggesting candidates for workshop participants. This board is chaired by Leslie Hogben, AIM Associate Director for Program Diversity. During fall 2009, HR Board members helped with the dissemination of the workshop Call for Proposals to professional organizations serving women and minority mathematicians and encouraged female and minority colleagues to consider submitting a proposal for an AIM workshop. After all the proposals were received in early November, members of the HR Board read and evaluated the proposals for diversity. The HR Board had a conference call in which the diversity and additional aspects of proposals were discussed. At the Scientific Board meeting in December, the HR Board chair conveyed the HR Board input.

AIM Directors continue to advise organizers in the planning of the workshop to best achieve the mathematical goal(s) set by the organizers. Part of this process involves the selection of invitees, and later applicants, and AIM Directors use the AIM Diversity Database to help the organizers broaden the impact of the workshop and be more inclusive.

The AIM Diversity Database is maintained and enhanced on an ongoing basis. During the past year substantial efforts have been made to add more minorities to the Database.

By reserving one quarter of the workshop places for applicants, an opportunity is provided for mathematicians who are not known to the organizers to participate. Deputy Director Estelle Basor and Associate Director Leslie Hogben use the AIM Diversity Database to recruit a diverse group of applicants for ARCC workshops. For each workshop, typically four to eight potential applicants are contacted and one or two apply (and are usually accepted). Some of the applicants recruited by AIM Directors have made significant contributions to the success of workshops. For example, Jean Clelland was encouraged by AIM to apply, and as described by one of the organizers, her contributions to the workshop "Nonlinear PDEs of mixed type arising in mechanics and geometry" had a major impact on the success of the workshop. She also helped organize the subsequent SQuaRE, "Higher Dimensional Embedding Problems for Riemannian Manifolds".

AIM's success in full inclusion of diverse groups, especially women and junior researchers, in AIM research workshops continues, as shown in Table 1.

Table 1. Participants in 2009 Workshops and SQuaREs*

	Women	Minorities	PhD after 2003	Reply rate
Workshops excluding REUF1 and Teachers' Circle	22.3%	4.4%	36.9%	86.7%
SQuaREs	14.9%	0%	23.4%	95.9%

***Notes:**

(1) Data from the August 2008 REUF1 workshop (see Section 6) and the June 2009 How to Run a Teachers' Circle workshop, which were funded by this grant, are excluded from the workshop data reported here because these were not standard research workshops (these workshops were much more diverse and the percentages for women and minorities in all workshops would be about four percentage points higher with the data from those workshops included).
(2) "Minority" means the NSF definition of under-represented minority, i.e., a US citizen or permanent resident who is a member of one of the following groups: Native American, Black, Hispanic, Pacific Islander.
(3) The % reported is the % of those who self-reported in the category among those who replied (those who declined to answer that question are included among the number who replied).
(4) The reply rate is the percent of participants who responded to the electronic request to answer questions, whether they answered a particular question or declined to answer.

SQuaREs is a new program, and AIM is working to develop strategies to improve inclusion of diverse groups of researchers in SQuaREs. AIM has worked for years to develop the methods used in managing "AIM-style" workshops, and both this experience and and the efforts of AIM Directors to enhance diversity through involvement in all aspects of workshop planning are illustrated by the higher rates of inclusion in workshops. New techniques are currently being developed for SQuaRES, For example, AIM Directors are encouraging successful and diverse subgroups of workshop participants to propose SQuaREs. The Scientific Board is also aware of this issue and is working with AIM Directors to better evaluate the diversity of SQuaREs proposals.

5.2. **New diversity and workforce initiatives.** One of the new initiatives to increase inclusion of under-represented groups, Research Experiences for Undergraduate Faculty (REUF), is described in Section 6. Here we briefly describe another new initiative.

At the SACNAS meeting in 2007 Prof. Aloysius Helminck of North Carolina State University described to the Math Institutes Diversity Committee a proposal he was developing to create a minority post-doctoral associate program, and requested the cooperation of the Institutes. Although the Institutes were supportive of the idea, a lack of coordination caused things to move slowly until a subcommittee of the Diversity Committee, led by Leslie Hogben of AIM, was formed at the 2008 SACNAS meeting to assist with the development the proposal. This subcommittee and additional AIM Directors worked with Helminck on the development of the proposal, which has now been funded by NSF. It is planned that each post-doc will spend a year at one of the Institutes, in addition to two years at a host university. AIM will also lend its workshop organizing expertise to the planning of workshops associated with the program.

In May 2009 Leslie Hogben of AIM and Chehrzad Shakiban of Institute for Mathematics and its Applications were elected co-chairs of the Math Institutes Diversity Committee.

6. Research Experiences for Undergraduate Faculty (REUF)

In October 2007 during a panel discussion at a conference for under-represented undergraduate mathematics majors, Prof. Roselyn Williams of Florida A & M University commented on the many wonderful research opportunities available to students today, and expressed the wish that faculty who devote themselves to teaching students could have a similar experience. Leslie Hogben, who was representing AIM at this conference, suggested to Williams that she propose an AIM workshop to provide an REU-type experience aimed at faculty at Historically Black Colleges and Universities (HBCUs) and other colleges where the faculty spend almost all their time teaching. As a result, Williams and Yewande Olobumu of Spelman College and Joe Omojola of Southern University New Orleans proposed the workshop Research Experiences in Linear Algebra and Number Theory for Undergraduate Faculty (now called REUF1). This workshop received enthusiastic support from the AIM Scientific Board, was funded by this grant, and was held in August 2008 at AIM. Four senior mathematicians, Hogben, Prof. Bryan Shader of the University of Wyoming, AIM Deputy Director Estelle Basor, and AIM Director of Programming David Farmer presented problems in linear algebra and number theory, and then worked with small groups to investigate these problems.

The faculty member participants all came from undergraduate institutions, most having limited research activity. Data on participants is shown in Table 2. The participants investigated problems suitable for subsequent research with undergraduates. Many of the participants have subsequently worked with undergraduates on related questions. One of the participants, Jillian McLeod, has developed a long term collaboration with Leslie Hogben.

Table 2. Demographic data for REUF1 participants*

Category	% in category	response rate
Participant is at HBCU	75%	100%
Participant is African-American	75 %	75%
Participant is female	62.5%	75%

*Notes: "Participants" for this workshop excludes the mathematical leaders. Race and gender data are self reported in response to the electronic survey. Whether a participant is at an HBCU was determined from the institutions that participants reported when registering.

After the success of this workshop, Williams and Hogben were awarded NSF grant DMS 0901523 Research Experiences in the Mathematical Sciences for Undergraduate Faculty to hold a second workshop and related follow-up activity. This workshop was held in July 2009 at AIM. Hogben, Prof. Nate Dean of Texas State University, Prof. Phil Kutzko of University of Iowa, and Prof. Kent Morrison of California Polytechnic State University presented problems in linear algebra, graph theory, and algebra, and then worked with small groups to investigate these problems. A proposal for funding for three annual programs (workshop and follow-up) has been submitted by Williams, Hogben, and McLeod.

7. Completed workshops

1. Noncongruence modular forms and modularity

August 17 to August 21, 2009
Organized by Winnie Li, Tong Liu, Ling Long, and Ravi Ramakrishna

Original Announcement: This workshop will explore the arithmetic and analytic properties of noncongruence modular forms and their potential applications. A special focus will be on the connection between Scholl representations attached to noncongruence cuspforms and automorphic forms by applying modularity lifting theorems.

By a theorem of Belyi, any smooth projective curve defined over a number field is isomorphic to a modular curve for some finite index subgroup of $SL(2,\mathbf{Z})$. The majority of these are noncongruence subgroups. For example, the degree 3 Fermat curve $E : x^3 + y^3 = 1$ is the modular curve for the degree 3 Fermat group Φ_3 , contained in $\Gamma(2)$. The space of weight 2 cuspforms for Φ_3, denoted by $S_2(\Phi_3)$, is 1-dimensional and generated by

$$\begin{aligned} f(z) =& q^{1/2} + ... + 70q^{5/2} + ... + 23000/3^2 q^{7/2} + ... + 6850312202/3^3 q^{13/2} \\ =& \sum a(n)q^{n/2}. \end{aligned} \tag{1}$$

Observe that the Fourier coefficients of f are rational numbers with unbounded denominators which indicates that Φ_3 is noncongruence.

On the other hand, the celebrated Taniyama-Shimura modularity theorem established by Wiles et al. says that the ℓ-adic representation attached to E comes from a weight 2 congruence normalized newform $g(z) = \sum b(n)q^n$. Atkin and Swinnerton-Dyer discovered remarkable congruence relations satisfied by the Fourier coefficients of noncongruence form f and congruence form g for almost all primes p:

$$a(np) - b(p)a(n) + pa(n/p) \equiv 0 \bmod p^{1+ord_p n}.$$

They further suggested that such three-term congruence relations on Fourier coefficients of noncongruence forms should hold in general for a basis depending on p with suitably chosen algebraic integers replacing $b(p)$ and p.

Major breakthroughs in the study of noncongruence cuspforms were achieved by A. Scholl. In order to understand the Atkin and Swinnerton-Dyer congruence relations, Scholl constructed a compatible family of $2d$-dimensional ℓ-adic Galois representations attached to each d-dimensional space of noncongruence cuspforms of integral weight $k \geq 2$ under general assumptions. The congruences above result from the Scholl representations attached to $S_2(\Phi_3)$ isomorphic to the ℓ-adic Galois representations attached to $g(z)$.

Proposed below is a partial list of topics to be discussed during the workshop. The participants are welcomed to comment on it and suggest related topics of their interests.

- When will 2-dimensional representations of the Galois group of a totally real field attached to noncongruence cuspforms arise from Hilbert modular forms?
- To what extent will the bounded denominator property on Fourier coefficients characterize a congruence modular form?

- Can the conductor of the Scholl representations be determined in terms of the data of the noncongruence cuspforms? If so, how to do it effectively? Can this be extended from $\mathbf{Q}$ to a totally real number field?
- Search for fast algorithms to enumerate noncongruence subgroups. As an application, one can determine the noncongruence subgroup of least index in the modular group having exceptional eigenvalue of the Laplace operator.
- What are the analytic properties of noncongruence Maass Waveforms? What are the distributions of their coefficients and the orders of their scattering matrices?

Organizer's report:

The *Noncongruence modular forms and modularity* conference was held at AIM from August 17-21, 2009. The goals of the conference were to report on recent progress in the field, identify areas for future research, both for short and long term projects, and begin work on (some of) these questions. Particular emphasis was placed on

- the recent progress and examples of Atkin, Hoffman, Li, Liu, Long, Verrill, and Hoffman's former REU groups in this area. Areas of focus were Atkin-Swinnerton-Dyer Congruences, Automorphy of Scholl representations attached to noncongruence cuspforms, and the Unbounded Denominator Conjecture,
- the links between the '$R = T$' modularity theorems on the congruence subgroup side and applications to noncongruence questions,
- computational questions involving specific examples, general questions such as how to describe a noncongruence subgroup computationally and the sort of data one might wish to amass in publicly available tables.

Introductory lectures were held in the morning and we broke into working groups in the afternoons. Participants were free to join the working groups. Each afternoon each working group briefed the audience what was done in the previous day so that people could decide which working group to join that day. In fact many people did switch groups.

The working groups were (roughly)

- Modularity
- Unbounded Denominator Conjecture (UBD)
- Atkin-Swinnerton-Dyer Congruences (ASD)
- Computational Methods

Our main goals were to formulate interesting questions and to start work on some of these. Listed below are the questions proposed during the workshop.

(1) For a finite index subgroup Γ of $SL_2(\mathbb{Z})$, denote by cd_Γ its congruence defect, that is, the index of Γ in the smallest congruence subgroup containing Γ. Thus Γ is a congruence subgroup if and only if $\mathrm{cd}_\Gamma = 1$. Let K be a number field. Does there exist a constant c_K such that for any smooth projective irreducible curve C defined over K, there exists a finite index subgroup $\Gamma \subset SL_2(\mathbb{Z})$ with $\mathrm{cd}_\Gamma \leq c_K$ so that there is a surjective morphism from the modular curve $X_\Gamma \twoheadrightarrow C$? When $K = \mathbb{Q}$ and C is

an elliptic curve, the Taniyama-Shimura modularity theorem says that we can choose a Γ with $cd_\Gamma = 1$.

(2) What properties do Scholl representations have that distinguish them among all l-adic representations?

(3) Let $H_5 := \left\{ \gamma \in SL_2(\mathbb{Z}) \mid \left(\frac{\eta^{12}(11_z)}{\eta^{12}(z)}\right)^{1/5} |_\gamma = \left(\frac{\eta^{12}(11_z)}{\eta^{12}(z)}\right)^{1/5} \right\}$. This is a noncongruence subgroup whose Eisenstein series are known to have algebraic Fourier coeffcients. Find a basis for weight 2 Eisenstein series for H_5 and search for ASD congruences.

(4) Can we say anything about modularity of semi-linear Galois representations in general? This is a vast generalization of the results in Atkin-Li-Liu-Long, where certain Scholl representations admitting quaternion multiplications defined over quadratic extensions of $\mathbb{Q}$ were studied.

(5) Study l-adic representations numerically associated to 2 families of abelian varieties over specific Shimura curves and compute the zeta functions.

(6) Is there any analogue of Hecke operators to decompose Scholl's representations? We need auxiliary operators to do that. That is why character groups are being considered first.

(7) Does a (restricted) 4-dimensional Scholl representation with image in GO(4) factor to reflect the fact that GO(4) is a quotient of GL(2) $\times$ GL(2)?

(8) Is it fruitful to study noncongruence modular forms via representation theory?

(9) Study the unbounded denominator (UBD) conjecture via representation theory.

(10) Study the unbounded denominator conjecture for generalized modular forms (GMF). Is there any connection between generalized modular forms and noncongruence modular forms? (During the workshop, Kohnen and Mason proved a theorem which reduced the UBD discussion of GMF to UBD of noncongruence modular forms in a special case.)

(11) Is there any good and effective method to find the conductor of Galois representations constructed from X_Γ, such as Scholl representations?

(12) Is it possible that a potentially "bad" prime on modular curves becomes a good prime for the associated Galois representation? How do we detect those primes?

(13) Study Poincare series and Kloosterman sums for noncongruence subgroups and modular forms.

(14) Study the field of definition F of X_Γ and find equations (effectively) to define the field F.

(15) Study Scholl's theory for triangle groups and their generalizations.

(16) Is it possible to define "CM" forms for noncongruence modular forms?

(17) Is it possible to recover the characteristic polynomial of the Frobenius by modular symbols? In particular, what about the case that Scholl representation is decomposable?

(18) Suppose that $\dim_{\mathbb{C}}(S_k(\Gamma)) = 1$. In this case the Scholl representations are modular. How does this relate to the Tate conjecture?

(19) Belyi's theorem says that a smooth irreducible projective curve C defined over a number field K is a modular curve X_Γ for infinitely many finite index subgroups Γ of $SL_2(\mathbb{Z})$. What are common properties among such Γ that realize C? Is there any best subgroup among them?

(20) Is there congruence relation on the Fourier coeffcients of non-cuspidal noncongruence form?
(21) Study the ASD congruence for Eisentein Series and super congruences.
(22) Compare the characteristic polynomial of Frob_p both for l-adic cohomology and p-adic cohomology over the modular curve X_Γ.
(23) Develop a software package that would compute the L-functions of Scholl's representations.
(24) Compute examples of these representations where the genus of the base curve is greater than 0, or is defined over a number field larger than $\mathbb{Q}$.
(25) Create a database of noncongruence subgroups of small index inside $SL_2(\mathbb{Z})$, including such data as: defining equation for modular curve and j-map, a basis of modular forms of low weight.
(26) Find examples of weight 1 cusp forms (do Tonghai Yang's examples coming from Fermat curves, satisfy ASwD congruences?)

For the *modularity* and *ASD* groups, various related problems were formulated and approaches discussed, but no progress was made.

The *UBD* group worked on the following problems related to the unbounded denominator (UBD) properties and more generally p-adic properties of noncongruence modular forms:

- Find the relations between noncongruence modular forms and generalized modular forms (GMF) defined by Knopp and Mason, especially their unbounded denominator (UBD) behaviors.
- Given a non-arithmetic subgroup $\Gamma of SL_2(\mathbb{Z})$, discuss to what extent a fruitful theory of p-adic modular forms on Γ can be developed. One might ask, for example, whether there are analogues for non-congruence subgroups of the well-known phenomenon that modular forms on $\Gamma_0(p)$ are p-adic limits of modular forms on $SL_2(\mathbb{Z})$. This question could be approached through a better understanding of the geometry of the rigid analytic space $X_\Gamma(C_p)$ together with its natural projection $X_\Gamma \to P_1$.

The following progress has been made:

- Computed a special case of the UBD behavior of GMFs which might be reduced to the UBD behavior of noncongruence modular forms.
- Identified a noncongruence modular form which seems to be a p-adic limit of congruence modular forms.

The core of the *computational* group consisted of Bryan Birch, David Farmer, William Hoffman, Jonas Kibelbeck, Chris Kurth, Ben Linowitz, Richard Moy, Frederick Strömberg, John Voight, and was joined on Friday by Tong Liu, Ron Livné, Ravi Ramakrishna, and Anthony Scholl, both of whom gave presentations. Each day there was a discussion about which problems to address, followed by some study of the problems. Some highlights:

(1) There were presentations by Hoffman, Strömberg, Kurth, Farmer about the computational work done so far.
(2) A list was drawn up of desiderata for software packages dealing with subgroups of finite index in $SL_2(\mathbb{Z})$. (see that attachment).

(3) John Voight gave a tutorial on his Magma package for computing fundamental domains and traces of Hecke operators of subgroups of $SL_2(\mathbb{R})$ defined by quaternion algebras.
(4) Frederick Strömberg worked on his algorithms for computing modular forms on Fuchsian subgroups of $SL_2(\mathbb{R})$.
(5) David Farmer computed the conductor and Euler factors for several of the L-functions of Scholl's representations that were computed experimentally by Richard Moy. He explained his method to the group.
(6) Tong Liu lectured on how to bound the conductors of Galois representations.
(7) A start was made on computing Atkin-Swinnerton-Dyer congruences for Eisenstein series.

2. Relative trace formula and periods of automorphic forms

August 24 to August 28, 2009

Organized by Herve Jacquet, Erez Lapid, and Akshay Venkatesh

Original Announcement: This workshop will be devoted to the study of the relative trace formula and periods of automorphic forms.

In particular, we hope to formulate a precise general conjecture for the exact value of period integrals which encompasses all known cases (either proven, e.g. torus periods on GL(2) (Waldspurger), unitary periods on the general linear group (Jacquet), or conjectural e.g. the work of Ichino and Ikeda on the Gross-Prasad period). The relative trace formula relates periods integrals on two different groups, and often reduces a "difficult" period integral to an "easy" one, thus providing a powerful tool to attack the putative conjecture. Thus far the study of the RTF has been primarily example-based, and we hope to (begin to) develop a general theory.

The main topics for the workshop are

- Examine all known cases of period integrals, with an eye towards formulating a general conjecture
- Formulation of a "general" relative trace formula and its "compatibility" with the general conjecture.
- Methodology of the relative trace formula: e.g. adapting the techniques of Ngo to relative setting, development of a local relative trace formula.
- Examine the compatibility of the general conjecture with "explicit constructions" e.g. backward lifting, Theta.
- Periods in the case where there is not multiplicity one (e.g. unitary and orthogonal periods on the general linear group).

Organizer's report:

Below is a list of the main themes of discussion in the conference.

One theme was a conjectural description of a formalism relating period integrals to the principle of functoriality and automorphic L-functions. Part of this effort, was the description of a general framework for the various relative trace formulas discovered so far. In particular, one of the goals was to arrive at a precise formulation of a general Gross-Prasad-Ichino-Ikeda type conjecture relating the square of the period integrals to special values of L-functions and local harmonic analysis. This is related to recent work of Sakellaridis-Venkatesh. A crucial remark was made for the seemingly trivial case where the period subgroup is the diagonal in a product. By considering L-packets, it became possible to fit this case in the general framework. There was also some discussion about the non-tempered case. Gross-Prasad Ichino-Ikeda formulations are not available when there is no local uniqueness. The lecture of Omer Offen shed light on the case of the unitary periods.

A related theme was the conjecture of Gross-Prasad for unitary groups and generalizations of the Gross-Zagier formula for unitary groups. Jacquet and Rallis have proposed a relative trace formula approach to the Gross-Prasad conjecture. The relevant fundamental lemma has been proved recently in positive characteristic by Zhiwei Yun. At the conference it was established that methods of logic can be used to extend this result to the case of characteristic 0 (and large residual characteristic). On the other hand, Shu-Wu Zhang has proposed a relative trace formula approach to the Gross-Zagier formula with possible generalizations to unitary groups in n variables. Again, a crucial step is a new kind of fundamental lemma where derivatives of orbital integrals appear. Preliminary work on the case of $n = 3$ was presented at the conference.

A third theme was the relation between the relative trace formula and the Weil representation. Zhengyu Mao gave a definitive exposition of the subject.

A fourth theme was the local theory. Fiona Murnaghan presented a lecture on (local) distinguished representations. In particular, she recalled the concept of a distinguished representation which is relatively cuspidal or relatively discrete series. This suggested revisiting the case of the symplectic group as a period subgroup. This also led to a question: what is the correct notion of an distinguished automorphic representation which is relatively cuspidal? Separately there was a preliminary discussion of the notion of a local relative trace formula.

Finally, there was a discussion of the relation with analytic number theory. This included the possibility of using the relative trace formula for obtaining estimates on L-functions. Some results in this direction were previously obtained by Ramakrishnan-Rogawski and Feigon-Whitehouse.

3. Permanents and modeling probability distributions

August 31 to September 4, 2009

Organized by Alon Orlitsky, Narayana Santhanam, and Krishnamurthy Viswanathan

Original Announcement: This workshop will study the problem of estimating a probability distribution from a small data sample it generates. The workshop will investigate consolidating a theoretical and algorithmic framework for this topic.

Aspects addressed will include distribution, probability, and population estimation, prediction, and classification. Emphasis will be on recent methods related to Good-Turing estimation and patterns, which cast the problem in a combinatorial and machine-learning perspective and relate it to integer and set partitions, symmetric polynomials over many variables, computation of matrix permanents, Markov chain Monte Carlo techniques, universal compression, and geometric programming.

Organizer's report:

The goal of the workshop was to further recent developments on estimating a probability distribution p over a countable *alphabet* $\mathcal{A}$ if we are allowed to sample with replacement from the distribution very few times. The workshop approached the problem mostly using the *pattern* technique, described below. Given a sequence $\bar{x} = x_1 x_2 \to x_n \in \mathcal{A}^n$ of symbols drawn from a discrete alphabet $\mathcal{A}$, the pattern of $\bar{x}$, denoted $\bar{\psi}(\bar{x})$, is the sequence obtained by replacing each symbol in $\bar{x}$ by its order of appearance. For example, the pattern of $\bar{x} =$ "*abracadabra*" is $\bar{\psi}(\bar{x}) = 12314151231$. The connection to permanents arises from the fact that when the sequence $\bar{x}$ consists of independent and identically distributed samples, the probability induced on the pattern can be written as the permanent of an appropropriately constructed matrix. In addition to patterns, the workshop also considered alternative approaches arising from the study of exchangeable random partitions and Bayesian estimation, the *large number of rare events* regime, Good-Turing estimation, and other divergence-based approaches.

The workshop featured talks on selected topics in the mornings followed by break-out sessions in which attendees split into groups to work on open problems. We first describe the talks presented and then the problems discussed in the break-out sessions.

Presentations

The presentations were on the following topics.

- An introduction to probability estimation and distribution modeling. Results derived through the pattern-based approach to probability estimation where the estimation quality is measured in terms of the *attenuation*—the rate at which we underestimate the true probability of the given sequence. These results were contrasted with those that could be obtained using extensions of the Laplace estimator.
- A more detailed lecture on probability estimation. The main focus of the lecture was a step-by-step exposition of the proof that pattern-based estimator achieves an attenuation of 1 for all observed sequences. Good-Turing and Laplace estimators were also analyzed.

- Distribution modeling where the goal is to estimate statistics that do not depend on the support set of the distribution. In particular, the pattern maximum-likelihood was considered. Since deriving the pattern maximum-likelihood is difficult in general, results on specific profiles were presented and more general results were conjectured based on them.
- A cross-section of algorithms and results on computing the permanent of a matrix, and the implications of these results to the problem of computing and thereby maximizing the probability of a pattern.
- Approaches to the probability estimation problem that are not based on patterns were also explored, including an introduction to exchangeable random partitions and a demonstration of the connection between patterns of *i.i.d.* processes and paint-box processes developed by Kingman. The talk also included subjects from Bayesian estimation suggesting the possibility that the pattern-based estimators could be written as a Baye's estimator.
- The *large number of rare events* (LNRE) setting. In this setting, all symbol probabilities are on the order of 1 over the sample size, and it captures the regime where the sample- and alphabet-sizes are comparably large. In particular, the talk focussed on estimators derived for the probability estimation problem in this setting.
- Additional talks on probability estimation based on the K-L divergence, Dirichlet processes, and sticky channels.

Open problems

The afternoon sessions were devoted entirely to open problems that either existed at the beginning of the workshop, or emerged as a result of talks and discussions. The first afternoon featured an open problem session where a number of interesting problems related to patterns and probability estimation were proposed. Subsequently the attendees split into smaller groups to tackle a subset of these problems. We describe the problems considered and the progress achieved below.

- Connections to Bayes estimation: Is it possible to recast the pattern-based diminishing attenuation estimators as a Bayesian estimators? and if not, do classical Bayesian priors on infinite alphabets, such as Dirichlet, result in a Bayesian estimator that achieves diminishing attenuation in the non-Bayesian setting? While the discussions during the workshop did not resolve the questions fully, the first question has been, interestingly, answered in the negative subsequent to the workshop. In addition, collaborations starting with the workshop have analyzed the attenuation of some Bayesian estimators, in particular, of the Ewens sampling formula.
- Consistency of pattern maximum-likelihood: Let $\hat{P}_{\bar{\psi}}$ denote the distribution that maximizes the probability of the pattern $\bar{\psi}$. It is known that this estimate is consistent for all discrete distributions over finite or countably infinite alphabets. However it is also known that no estimator is uniformly consistent for this class of distributions. Therefore a weaker uniform consistency result was desired. The following set-up was examined. For any positive integer k, let $\mathcal{P}(k)$ denote the set of all discrete distributions with support of size at most k. Let $||\cdot||_1$ denote the ℓ_1 distance. Let X^n denote an *i.i.d.* sequence of random variables, $\bar{\psi}(X^n)$ denote its pattern. For any

non-decreasing function $f(n)$, the pattern maximum-likelihood is said to be uniformly consistent for $\mathcal{P}(f(n))$ if

$$\lim_{n\to\infty} \sup_{P\in\mathcal{P}(f(n))} Pr\Big(||P - \hat{P}_{\bar{\psi}}(X^n)||_1 \geq \tau\Big) = 0.$$

From results on property testing in Computer Science, it was observed that such a result is unlikely to be true for $f(n) = \Omega(n \log n)$ and is likely to be true for $f(n) = \mathcal{O}\Big(\frac{n}{\log n}\Big)$. Some progress was made in resolving the residual ambiguity by applying techniques used for the probability estimation problem in the LNRE regime.

- Two-dimensional patterns: How does one handle memory? To this end, candidates were proposed for two-dimensional patterns that would model dependencies between two random variables. Discussions during the workshop revealed that even elementary computations in this scenario—for example, the maximum probability of patterns obtained by sampling a pair of random variables (X, Y) thrice—was non-trivial. However, discussions that continued after the workshop among the participants have resolved the elementary problems posed in the workshop, and we have moved into thinking about extending this framework in more detail.
- Largest alphabet size of pattern maximum-likelihood estimate: For a given pattern $\bar{\psi}$ let $\hat{P}_{\bar{\psi}}$ denote the distribution maximizing the pattern probability. For any distribution, P let $S(P)$ denote its support size. It was conjectured based on certain patterns whose $\hat{P}_{\bar{\psi}}$ can be derived in closed form, that for all patterns if $\hat{P}_{\bar{\psi}}$ is discrete, then

$$S(\hat{P}_{\bar{\psi}}) \leq \frac{m^2}{2}.$$

Attempts were made to prove this conjecture during the break-out sessions and some promising approaches were uncovered.

- Computation of pattern maximum-likelihood: Discussions were held on efficient ways to compute the probability of a pattern and thereby the pattern maximum-likelihood. These focussed on improving the existing Markov chain monte carlo techniques that are employed as for computing pattern probabilities in an expectation-maximization algorithm designed to compute $\hat{P}_{\bar{\psi}}$.

4. Convex algebraic geometry, optimization and applications

September 21 to September 25, 2009
Organized by William Helton and Jiawang Nie

Original Announcement: This workshop will be devoted to the study of "Convex Algebraic Geometry" and some of its numerous applications. Convexity plays a fundamental role in mathematics, and its ubiquity in optimization makes it of crucial importance in many domains of application. In such situations, the geometric properties of convex sets are complemented by additional algebraic structure (e.g., the semialgebraic case, where sets are defined by means of polynomial inequalities). In this case, the rich interactions between the geometric, algebraic, and computational aspects are not yet well-understood.

Falling into this setting are classical linear programming (LP), the more recent area of semidefinite programming (SDP), and the associated linear matrix inequalities (LMI),which have had a major impact on engineering systems, combinatorial optimization and other areas. One focus of the workshop is the study (arising from linear systems engineering) of polynomials in matrices whose form does not depend on the size of the matrices; this requires development of a noncommutative semialgebraic geometry.

Convex Algebraic Geometry involves a healthy combination of real algebraic geometry, functional analysis, operator theory, convex optimization and several areas of application. We expect a diverse group with common emerging interests.

Organizer's report:

We start with a little bit of terminology. A linear pencil is a function $L : \mathbb{R}^g \to$ $d \times d$ *symmetric matrices* of the form

$$L(x) := L_0 + L_1 x_1 + \cdots + L_g x_g$$

with the L_j being $d \times d$ symmetric matrices. A linear matrix inequality (LMI) is one of the form

$$L(x) \succeq 0$$

with $\succeq 0$ meaning $L(x) \succeq 0$ is positive semidefinite. LMIs have been one of the main successes and interests of optimization theory and numerics in the last 15 years. The study and algorithms for solution of LMIs is called semidefinite programming. A polyhedron is the set of solutions to ordinary linear inequalities and by analogy the set of solutions to an LMI is called a spectrahedron. That is, it is a set in $\mathbb{R}^g$ of the form $\mathcal{D}_L := \{x \in \mathbb{R}^g : \ L(x) \ is \ PSD \ \}$; obviously it is convex and semialgebraic. If a set $\mathcal{C}$ in $\mathbb{R}^g$ is a spectrahedron, it is said to have an LMI representation.

The following is an account of some of the group topic discussions which occurred each afternoon:

- **Dimensional differences of faces of nonnegative and SOS polynomials**
  The cone of nonnegative polynomials contains the cone of polynomials which are sums of squares of polynomials (SOS) properly. Their geometric structures are very different. It is known that there are significantly many more polynomials than sum of squares. This can be reflected by the dimensional differences between their faces. Their dimensions are different when there are nonnegative polynomials which are not

SOS. These dimensional differences can bring valid inequalities to separate nonnegative polynomials from the SOS cone. The main research issue is how to get these separation inequalities efficiently.

- **SDP representation of convex sets and hulls**
The main topic of this workshop is SDP representability issues. There are recent constructions by Lassere and Parrilo parameterizing convex semialgebraic sets as projections of linear matrix inequalities. Helton-Nie proved this happens when the boundaries are smooth and have positive curvatures. The main research issues now are how to handle the singularities and zero curvature. To resolve singularities, the standard techniques from algebraic geometry are potentially very useful, but the biggest trouble is how to preserve convexity. For the case of zero curvatures, the difficulty is how to represent nonnegative polynomials, which is one of the most difficult problems in polynomial optimization. A lot of trial approaches are discussed in the group discussion.
A related issue is how to represent the convex hull of varieties. When a variety is a curve parameterized rational polynomials, its convex hull has an easy SDP representation. However, for higher dimensional varieties, its lifted LMI is quite difficult to get. The main issue: there is no general SOS type representation for nonnegative polynomials.

- **Numerical methods in polynomial optimization**
The main discussion topic was how to solve large scale polynomial optimization. For SOS relaxations, if interior-point methods are used, only small problems could be solved. The reason is the resulting SDP from SOS relaxation is in the most difficult class of SDP problems, because the number of equality constraints is the square of matrix length in magnitude. For this type of SDP problem, the regularization type of methods like augmented lagrangian are very suitable. The memory requirement is very low. The main issues in this type methods are how to solve a subproblem efficiently. Since the performance of Conjugate Gradient (CG) iteration significantly effects the convergence of the whole algorithm, the preconditioning is important. The future research problems include: how to improve semismooth Newton method, find better preconditioners for CG iterations in solving Lasserre's relaxation, seek more clever line search techniques, find better ways to update parameters. Nie described his new conjugate gradient SOS solver and empirical finding that positive degree 6 polynomials in higher dimension are often not SOS.

- **Quadratic modules and tropical varieties**
The discussion group originated from issues raised by Salma Kulmann on quadratic modules when Bernd Sturmfels pointed out they were related to the rapidly moving are of tropical geometry. This group consisted of about a dozen of workshop participants, at least half of them were young researchers. We first recalled the definition of the locally convex topology on the polynomial ring, and reviewed Berg's theorem which establishes that the quadratic module of sums of squares (SOS for short) is closed. This theorem was generalized later by Kuhlmann-Marshall to a finitely generated quadratic module M that describes a cone with open interior, and let to the key notion of stable quadratic modules due to Powers-Scheiderer. A key lemma in the proof is the fact that if a polynomial $p(x)$ is nonnegative on a cone, then so is

its leading term (homogeneous component of highest degree) $LT(p(x))$. Salma suggested that $LT(M) := \{LT(p(x)) : p \in M\}$ is an interesting object for investigation, possibly under other weighted degree or monomial orderings w. (This should also connect to the "preodering membership problem" studied by D. Augustin in her dissertation.) Graduate student Cynthia Vinzant has a first result in this direction. She showed that given an ideal I, if for some weighted degree w we have that $LT_w(I)$ is real radical, then SOS + I is stable. This is related to work of Tim Netzer (dissertation). B. Sturmfels explained in this context the notion of Groebner Fan of an ideal. This led to a lively discussion. The following problems were suggested for further study:

Problem 1: Study the Groebner Fan of a quadratic module (Kuhlmann).

Problem 2: Study the examples of stable preorderings of planar curves described by Scheiderer in light of Vinzant's result (Marshall).

Problem 3: Given the quadratic module generated by a single polynomial in 2 variables, decide whether it is stable or not (Marshall). Before parting, Bernd Sturmfels gave us the definition of the tropical variety associated to an ideal, and provided some homework on this topic for the participants.

- **Degree Bounds for Positivstellensatz, boundary structure of Hyperbolic cones, derivative cones, extreme rays of hyperbolic cones**

  The discussion started with Markus Schweighofer describing the current state-of-the-art for the degree bounds for Positivstellensatz. Markus gave a sketch of some of the proofs and described the ingredients of the others. Then a discussion of the quality of these bounds and their limitations took place. A brainstorming activity on SDP relaxations and convex hulls lead the group into the subject of hyperbolic polynomials and their hyperbolicity cones (also called hyperbolic cones). Levent Tunçel first went over the structural results by Renegar involving the derivative cones and then discussed some newer, unpublished results expressing every hyperbolic cone precisely as the intersection of its derivative cones (where the intersection ranges over all interior points of the original hyperbolic cone). This result shows that the boundary structure of hyperbolic cones is completely determined by its derivative cones. The session ended with a discussion of hyperbolicity preserving operations on polynomials, strictly hyperbolic polynomials, some partial characterizations of the extreme rays of hyperbolic cones and finally, possible future research directions for improving our understanding of the boundary structure of hyperbolic cones.

- **Functional Analysis Issues**

  One session was attended by Salma Kuhlmann, Bill Helton, Victor Vinnikov, Mihai Putinar and graduate students Jeremy Greene, Martin Harrison, Chris Nelson, and Joules Nahas.

  We consider Salma's question of investigating the closure of the SOS polynomials in various locally convex topologies. These closures would then give a quantitative measure of how the topologies differ. In particular, the SOS polynomials are already closed in the finest locally convex topology; and, as it figures importantly in the work of Laserre, the closure in the L1 (on the coefficients) topology is the polynomials which are non-negative on the unit cube.

  The session began with Salma - with her background in model theory and semi-algebraic geometry - exposing the relevant objects and topologies to the functional analysis who made up the remainder of the group. We then generated several conjectures, including the guess that the closure of the SOS polynomials in g variables in the L^2 (on the coefficients) topology would yield exactly f in H^2 of the g-polydisc whose restriction to the cube of $[0,1)^g$ are non-negative.

We also heard reports from each of the graduate students at UCSB and UCSD and discussed possible future directions for their research. Martin Harrison and Joules Hahas developed a non-commutative version of Laserre relaxation; and Joules proved and used smoothness results for PDEs to established an improved version of Schmudgen's positivstellensatz for the Weyl algebra. Jeremy Greene has characterized non-commutative plurisubharmonic polynomials; Christopher Nelson has characterized all non-commutative harmonic polynomials.

Another session looked at the basic question is what are the corners of a spectrahedron. That is, what is the smallest rank r_* of a matrix of the form $L(x)$ with $x \in \mathcal{D}_L := \{x : \ L(x) \succeq\}$? What ranks occur? These questions were featured in Bernd Sturmfel's talk which opened the conference. See slides of Sturmfels talk posted on the AIM website for definitions pictures and context. Another issue is how might one possibly compute x_* minimizing rank $L(x)$. Rank minimization subject to convex constraints is an important practical but highly nonconvex problem. Maryam Fazel with Parrilo and Recht have a probabilistic analysis for minimizing *trace* $L(x)$ as a heuristic for finding an x_*, which is analyzed in a much more special case. This parallels the famous analysis by Candes and Tao. A session discussion speculated on a relaxation of her method (with Parrilo and Recht) dependent on a parameter ϵ, and started to see if one could prove probabilistic estimates of success by generalizing her methods. We looked at polytopes for starters.

In addition there was a session was on matrix completion problems arising from applications; notably one from statistics and one from engineering systems theory. These were suggested by Carolyn Uhler and Parikshit Shah graduate students in UCB Statistics Dept and MIT Engineering respectively. Any advance on the basic classical matrix completion problems would increase the range of results in these areas. Parikshit described a clever generalization via matrix completion of the crucial (to control) Finsler Lemma. Carolyn gave elegant results on covariance matrices. Also arising from the statistics application there is an interesting extra constraint one can add to the completion problem, thereby generating more open questions. The pure math question is:

We are given a matrix X of size $m \times n$ ($n < m$) and an undirected graph G on m vertices. We denote by $A(G)$ a G-partial matrix, where the entry a_{ij} is defined if and only if $(i,j) \in E(G)$ or $i = j$. Under what conditions on the row-vectors of the matrix X (respectively on the angles between the row vectors of X) does the G-partial matrix $XX^T(G)$ have a PD completion?

- **Sum of Squares and Polynomial Convexity**

  This session focused on the interplay between the concept of sum of squares (SOS) decomposition and the question of deciding convexity of polynomials. As defined by Helton and Nie, a multivariate polynomial $f(x) = f(x_1, \ldots, x_n)$ is SOS-convex if its Hessian $\nabla^2 f(x)$ is an SOS-matrix; i.e., if $\nabla^2 f(x)$ can be factored as $\nabla^2 f(x) = M^T(x)M(x)$ with a possibly nonsquare polynomial matrix $M(x)$. This is a sufficient condition for convexity of polynomials that can be efficiently checked with semidefinite programming.

Amir Ali Ahmadi an engineering graduate student at MIT, observed that there are other natural definitions that could be given for SOS-convexity. Namely, instead of working with the Hessian matrix, we can use the sum of squares relaxation on the inequality in the definition of convexity or in its first order characterization. He showed that all of these relaxations will be exactly equivalent. He then proceeded to prove that SOS-convexity is *not* a necessary condition for convexity of polynomials. He presented an explicit example of a trivariate homogeneous polynomial of degree eight that is convex but not SOS-convex.

Finally, he presented some more recent results, where there is much room for further research and new contributions. We shall end this discussion by sharing with the reader two related open problems raised by Amir that might be of interest.

After finding the first example of a polynomial that is convex but not SOS-convex, it is natural to ask in what degrees and dimensions there is a gap between convexity and SOS-convexity. Amirs recent results seem to suggest that there is a gap between convexity and SOS-convexity *exactly* in situations where there is a gap between nonnegativity and SOS (as described in the so called "Hilbert table"). Is this a mere coincidence? Or is there a deeper connection between some algebraic/geometric aspects of convexity and positivity that we currently do not fully understand?

The second problem also has to do with the connection of convexity/SOS-convexity with nonnegativity/sum of squares. Is it true that every convex nonnegative polynomial is a sum of squares? Although one might see no reason for this to be true at first glance, Amir has sensible evidence to indicate that this could potentially be true. About 10 days after the AIM workshop Greg Blekherman, active in many of our AIM discussions, posted an asymptotic estimate implying the answer is no for high degree polynomials. http://front.math.ucdavis.edu/0910.0656.

5. Rational curves and A1-homotopy theory

October 5 to October 9, 2009
Organized by Aravind Asok and Jason Starr

Original Announcement: This workshop will be devoted to studying recent interactions between rational connectivity and the newly developing theory of A^1-algebraic topology.

A smooth proper variety over an algebraically closed field k having characteristic 0 is rationally connected if any pair of k-points is contained in a rational curve. Rationally connected varieties have highly non-trivial arithmetic structure. More recently, rationally connected varieties have been shown to have interesting homotopic structure as well: they are connected (in an appropriate sense) from the standpoint of A^1-homotopy theory. One expects that techniques of homotopy theory can be applied to study arithmetic and geometry of rationally connected varieties and, conversely, geometric properties of specific rationally connected varieties can provide insights about A^1-homotopy theory. The goal of this program is to introduce participants studying arithmetic of rationally connected varieties to the techniques of homotopy theory and vice versa.

During the workshop, we will focus on the following problems.

- What is a good definition of rationally simply connected variety? Rational simple connectedness is a notion suggested by Mazur and developed by de Jong and Starr to prove existence of rational points for varieties defined over the function field of a surface. While various tentative definitions have been suggested, it is widely agreed that a "good" definition has not yet been found. Since rationally connected varieties are A^1-connected (in an appropriate sense), one can try to connect notions of higher rational connectivity with A^1-homotopy groups to obtain a "good" definition.
- Can weak approximation be approached using obstruction theory? A Serre fibration of CW complexes has a section only if certain obstruction classes vanish. These are defined in cohomology of the base having coefficients in homotopy of the fiber. Similarly, weak approximation of adelic points of a variety over a number field holds only if the Brauer-Manin obstruction vanishes (for rationally connected varieties this is conjectured to be the only obstruction). One hope is that A^1-homotopy theory will bridge these two results and extend them to answer the following geometric problem of Hassett and Tschinkel: Given a family of rationally connected varieties defined over a curve, are formal points weakly approximated by rational points?
- Can techniques of rational curves be used to understand norm varieties? A fundamental step in the proof of the Bloch-Kato conjecture by Rost and Voevodsky involves the construction of certain algebraic varieties with prescribed cohomological properties. In cases where these varieties can be explicitly constructed, one may use them to provide examples of unirational non-rational varieties generalizing the examples of Artin and Mumford. In the known approach to construct the norm varieties by induction on the length of the symbol, it is an observation that starting with the "usual" Brauer-Severi variety for a symbol of length 2, one gets a rationally connected variety. There are open questions related to this construction. For instance, can one construct a norm variety with the property that existence of an F-point (any field F) implies all the F-points are rationally chain connected? For norm varieties at the prime 2, Pfister quadrics provide an affirmative answer. For norm varieties at odd

primes, the question is open. We will explore norm varieties in greater detail with particular attention to their rational connectedness properties.

Organizer's report:

This workshop brought together people working around the theory of rationally connected varieties (in particular, aspects related to existence and density of rational points, and higher notions of rational connectivity), and people working in A^1-homotopy theory and related "motivic" aspects of algebraic geometry. Introductory lectures providing some common ground for the participants were given by Hassett (rational connectivity), Levine (A^1-homotopy theory), Haesemeyer (splitting varieties) and Colliot-Thélène (obstructions for rational points). Research talks were given later in the week by Esnault, Voisin, Krashen, Xu, DeLand, and Doran. Subsequent discussion focused on cohomological aspects of rationally connected varieties, obstructions for existence of rational points (especially using ideas from obstruction theory), and geometry of splitting varieties (generic and p-generic splitting varieties, and various generalizations).

- Obstructions to the existence of rational points: The participants tried to use the A^1-fundamental (sheaf of) group(s) to recover old and produce new obstructions to existence of rational points. Non-abelienness and lack of base-point produces complications, but new ideas were discussed and some concrete questions for G-torsors were raised.
- In order to study the obstructions suggested in the previous point, further computations of the A^1-fundamental group are required. "Generators and relations" presentations of the A^1- fundamental group were discussed. Known computations for rational surfaces, toric varieties and closely related "cellular" varieties were mentioned, with the aim of computing some obstructions.
- It is known that rationally connected varieties have no non-trivial pluri-differential forms. The converse was investigated. In particular, it was asked whether one could prove directly that a variety without pluri-differential forms had a trivial fundamental group. This problem was studied in the context of the Shafarevich conjecture, and also related to some work of Gromov on varieties with "large" fundamental group.
- The definition of rationally simply connected varieties was reviewed. In particular, the question "which geometric and arithmetic properties should characterize rational simple connectedness?" was discussed in detail. Examples of such properties include existence of points over function fields of surfaces, weak approximation, etc.
- The birational geometry of norm varieties and some generalizations was discussed. New kinds of splitting varieties for cohomological operations were constructed. For example, generalizing the notion of a norm variety, participants studied the problem of existence of varieties having a rational point if and only if a Massey product in étale cohomology exists. Such varieties exist and can be realized rather explicitly in terms of bundles of Brauer-Severi varieties. Wickelgren will investigate this further.
- Colliot-Thélène and Voisin observed that a birational invariant studied by Voisin was the same as an alternative invariant studied by Colliot-Thélène and Ojanguren.
- The Russell cubic threefold was investigated in detail. In particular, the A^1-homotopy theory of this variety (e.g., its A^1- connectedness properties) and its motive were studied.

6. Higher Reidemeister Torsion

October 19 to October 23, 2009

Organized by Sebastian Goette, Kiyoshi Igusa, and John Klein

Original Announcement: This workshop will focus on connections between different constructions for invariants of fiber bundles.

In the last fifteen or so years, certain invariants for fiber bundles have been constructed in three different ways: analytically, homotopy theoretically and Morse theoretically. Depending on how the theory is presented, the invariants take value in either the cohomology of the base space, or in some version of higher algebraic K-theory. We group these invariants under the collective name, higher torsion.

This workshop will focus on the various kinds of the higher torsion. At the current time it is not yet known whether the different approaches lead to the same invariant; this is probably the most fundamental question in the subject. Therefore, we hope to bring together geometrically minded analysts and topologists with the hope of developing a common language.

More precisely, the workshop will focus on

- the problem of whether the three approaches coincide
- axioms for higher torsion
- the translation from the analytical to the topological approaches
- applications to various kinds of moduli problems.
- classification of fibre bundles and topology of Diff(M)

Organizer's report:

Original goals. One of the main original goals of this conference was to discuss the central question of whether the three definitions of higher Reidemeister torsion agree. These are

- The Morse theory approach of Igusa and Klein
- The A-theory approach of Dwyer-Weiss-Williams
- Bismut-Lott analytic higher torsion

However, by the time the conference met, this had already been accomplished by major works of Badzioch, Dorabiala, Klein, Williams and Goette.

The other original goal of the conference was for the three camps to meet to explain to each other their respective techniques, in particular details of the three definitions of higher torsion.

The meeting. At the AIM meeting there were lectures on the first three days explaining the three different approaches to higher torsion by the experts in those fields including two of the organizers and two participants. On the first days, we made a list of questions, easy and difficult which we used later in the week as topics for group meetings. Two particular topics which were discussed thoughout the meeting were the questions *What does it mean for the higher torsion to be zero?* and *How does one show that the different approaches to Hatcher example are equivalent?* On the last three days we broke up into small groups to discuss these educational and research topics. The group discussions were very productive and made unexpected amounts of progress in the two hours that they met. We were all very pleased with these small group meetings.

Lectures. *Day one:* Sebastian Goette gave an overview of higher analytic torsion with a gentle lecture which avoided many technical details. Bruce Williams gave an overview of Dwyer-Weiss-Williams torsion concentrating on the homotopy torsion and the smooth torsion.

In the afternoon, we compiled the first list of questions and Bernard Badzioch, Sebastian Goette and Kiyoshi Igusa gave short accounts of DWW torsion and IK torsion answering the question: *What is the input and output of the higher torsion?* On each subsequent day we had two lectures in the morning and discussions in the afternoon. Small group meetings started on Wednesday.

Day two: Lectures by Bernhard Badzioch on an alternative construction of the Dwyer-Weiss-Williams torsion and by Kiyoshi Igusa on the definition and calculation of Igusa-Klein torsion. In the afternoon, we discussed Hatcher's example.

Day three: Sebastian Goette talked about Bismut-Lott torsion and Stacy Hoehn talked about the fiberwise end obstruction.

Day four: Ulrich Bunke talked about analytic torsion and its relation to differential K-Theory and Kiyoshi Igusa explained his axioms for higher torsion.

Day five: Wolfgang Steimle spoke about new geometric problems related to classical Whitehead torsion. We had discussions in the morning in lieu of a second talk.

On these days, small group discussions addressed the questions of:

- How to generalize the Axioms to the case of nontrivial coefficients
- What is Waldhausen's algebraic K-theory?
- Comparison of different approaches to the Hatcher Example.
- What does it mean if the higher torsion is zero?
- What is a flat superconnection?

7. Implementing algebraic geometry algorithms

October 26 to October 30, 2009
Organized by Hirotachi Abo, Anton Leykin, Sam Payne, and Amelia Taylor

Original Announcement: This workshop will be devoted to developing three packages,

- algebraic statistics,
- numerical algebraic geometry,
- toric algebraic geometry,

for the computer algebra system Macaulay 2. Macaulay 2 is a widely used computer algebra system for research and teaching in algebraic geometry and commutative algebra and is one of the leading computer algebra programs for performing such computations.

These three topics are all very active areas of research in computational algebra and algebraic geometry and are linked in surprising ways which lends them nicely to be the three packages of focus for this workshop.

- Algebraic Statistics: Some of the key varieties arising in the application of algebra and geometry to phylogenetics are toric, while other challenges in studying both phylogenetics and reverse engineering of biochemical systems are rooted in the need for better numerical techniques for algebraic geometry. It is also the case that solving such problems, and related problems more broadly in algebraic statistics, often require non-standard approaches to computing primary decompositions and other standard algebraic objects for which broadly available code might allow form greater experimentation and study.
- Numerical Algebraic Geometry: While there are tasks best accomplished numerically and other tasks that can be approached only symbolically, there is a multitude of problems in computational algebraic geometry currently unsolved by either. A system which allows a user to seamlessly access both
  the numerical and symbolic algorithms and to write hybrid programs will make possible the kind of experimentation that might solve these problems. Developing the ability to create hybrid programs is the primary
  focus of this package. Developing such a package requires a combination of a clear understanding of both numerical methods and current problems in
  algebra and geometry that might benefit from this package, like algebraic statistics and toric algebraic geometry.
- Toric Geometry: Toric geometry stands at the interface between commutative algebra, combinatorics, and geometry and has a rich history as a testing ground for emerging theories and general conjectures in algebraic geometry. Several topics of current research are suitable for computational exploration, and access to efficient software could lead to rapid and significant progress on open problems, including determining whether iterated normalized Nash blowups resolve arbitrary singularities and computing large sets of examples of normalized Nash blowups of higher dimensional toric varieties, computing weighted Ehrhart series, and implementation of algorithms in toric intersection theory.

Organizer's report:

Introduction. *Macaulay 2*, developed by Dan Grayson and Mike Stillman, is one of the leading computer algebra programs for research and teaching in algebraic geometry and commutative algebra. *Macaulay 2* has a structure called a package, which is designed to encourage, ease and expand contributions to *Macaulay 2*.

The main goal of this workshop was to write *Macaulay 2* packages implementing algorithms in the following rapidly growing areas:

- algebraic statistics (AS);
- numerical algebraic geometry (NAG);
- toric geometry (TG).

Here "writing a *Macaulay 2* package" means not only implementing related algorithms but also writing the corresponding documentation and tests for the code.

The above-mentioned areas have significant applications in fields as diverse as computational biology, coding theory, statistics and string theory. We hope, therefore, that the packages developed through this workshop will advance our knowledge and understanding across multiple disciplines. In the following paragraphs, we summarize the activities and achievements during the workshop.

Activities. On the first day, after presentations designed to develop a list of goals, we broke into three groups AS, NAG and TG. Due to the considerable overlap between these three area, there were interactions between the groups, but the members of each group generally stayed fixed during the workshop week. We started each morning session with a progress report from each group with members from other groups often asking questions and making suggestions. After the progress reports participants worked in groups on packages, consulting with other members of the workshop as needed. At the beginning of the afternoon session, we discussed various *Macaulay 2* related issues including "How-To" sessions. The items discussed during that period of time were recorded and posted at the workshop wiki. The wiki page can be found at the following URL:

`http://wiki.macaulay2.com/AIM2009/`

The remaining afternoon session was devoted writing packages with workshop members often working quite late. Below we highlight important events that occurred during the workshop.

As we mentioned, the morning session of the first day of the workshop was aimed at formulating a list of problems in each of the three areas of focus. There were introductory talks by Brandilyn Stigler and Sonja Petrovic (AS), Diane Maclagan (TG) and Anton Leykin (NAG). The presentations identified *Macaulay 2* packages which exist, but need improvement and the speakers suggested the creation of the following packages:

Algebraic Statistics

(1) studying connectivity of fibers of a map associated to varieties in statistical applications with the gaol of extending our ability to study such varieties;
(2) implementing Luis Garcia's `Singular` library "Markov.lib" in *Macaulay 2* and extend it to more graph types and for Gaussian models;
(3) identifiability of parameters in an algebraic statistical model;
(4) computing Markov subbases;

Numerical Algebraic Geometry

(1) solving zero-dimensional polynomial systems;
(2) tracking user-defined homotopies;
(3) positive-dimensional solution sets: irreducible and primary decompositions;
(4) Schubert calculus applications such as the Littlewood-Richardson homotopy and the Pieri homotopy;
(5) recovering exact data from numerical approximations and certification;
(6) interfacing with other numerical polynomial homotopy continuation software;

Toric Geometry

(1) toric sheaf cohomology;
(2) intersection theory;
(3) resolution of singularities;
(3) Nash blow-up;
(4) rings of piecewise polynomials;
(5) maps between toric varieties.

In the afternoon session of the fourth day of the workshop, each group reported their progress and demonstrated the packages resulted through the workshop. The AS group presented the new package `ToricFibers` and commented on the progress of two other new packages, `Graphs` and `GraphicalModels`. The NAG group displayed their new package `IdealOfGenericPoints` package and updated the group on progress made in the package `NumericalAlgebraicGeometry`. Finally, the TG presented their new package `ToricResolve` designed to perform resolution of singularities of toric varieties.

At the final morning session, each participant proposed their short-term and long-term goals related to the projects during this workshop (an activity found to be useful by the organizers at other workshops). Some of the projects initiated at this workshop will be discussed and explored further at a *Macaulay 2* workshop at MSRI, January 8–12, 2010.

Achievements

Algebraic Statistics

- Given a design (dxn)-matrix A of a normal toric model the new package `ToricFiber` provides functions for building and working with a graph describing the fiber for all possible vectors b in the cone(A) such that b is a linear combination of at most d columns of A. Such fibers play a key role in one possible algorithm for computing minimal generators of a toric ideal and may lead to a more efficient way of studying the model rather than computing the full minimal generating set. The package allows users to (a) study if there is yet a more efficient algorithm for computing these toric fibers and (b) if there is a special structure in the case of statistical models that allows even more efficient construction of the information required to study the models.
- A rough package for working with discrete models described by conditional independence statements represented by a directed acyclic graph `Markov` was implemented by Luis Garcia in both *Macaulay 2* and *Singular* — which often given different answers to problems. The group determined why different answers occur, developed a new

package `Graphs` to make uniform the use of graphs by several packages, including this one, and began writing code to include undirected graphs and Gaussian models. Finally, the package was renamed `GraphicalModels`.

Numerical Algebraic Geometry

- Numerical Schubert Calculus: Numerical Schubert Calculus is used to numerically solve problems in enumerative geometry. The NAG group worked on the `NumericalSchubertCalcul` package that uses the Littlewood-Richardson homotopy and the Pieri homotopy.

  The Littlewood-Richardson (LR) homotopy in PHCpack was recently released as version 2.3.52 (PHCpack is a general-purpose solver for polynomial systems by homotopy continuation). The *Macaulay 2* package `LRhomotopies`, which provides an interface to the LR homotopy in PHCpack, was completed.

  The function `solveSimpleSchubert` uses the Pieri homotopy to find all solutions to simple Schubert problems. It uses algorithms described in the paper "Galois groups of Schubert problems via homotopy continuation", Mathematics of Computation, 78 (2009) 1749–1765, by Anton Leykin and Frank Sottile.
- Interfacing with PHCpack: In addition to LRhomotopies.m2, the multiprecision root refiners of PHCpack were also improved during the workshop and used to compute accurate generic points via the `NumericalAlgebraicGeometry` package in *Macaulay 2*.
- Recovery of exact data from numerical approximations: One of the advantages of numerical algebraic geometry is that it is simple to produce approximations of generic points on any given algebraic variety. The NAG group worked on a package called `IdealOfGenericPoints` which recovers exact defining equations for the variety from the approximated points.
- The NGA group started to build the foundation for dealing with the positive-dimensional case. In particular, the `WitnessSet` type has been contructed.
- The package `NumericalAlgebraicGeometry` was recently submitted to the Journal of Software for Algebra and Geometry (`http://j-sag.org`) for publication.

Toric Geometry

- The TG group worked on code for the `NormalToricVarieties` package, including developing intersection theory, implementing resolution of singularities (`ToricResolve`), and interfacing with the `Polyhedra` package. After its initial implementation, they were able to speed up the resolution of singularities by a factor of ten and discussed possibilities for future improvement.
- Some members of that group also worked on the package `ToricVectorBundles` which was recently submitted to the Journal of Software for Algebra and Geometry (`http://j-sag.org`) for publication.

Closing comment. Future work on packages will be recorded in a series of wiki pages that were planned at the AIM workshop. Those interested should watch the *Macaulay 2* page

`http://www.math.uiuc.edu/Macaulay2/`

and the *Macaulay 2* google group.

8. The Cuntz semigroup

November 2 to November 6, 2009

Organized by Nate Brown, George Elliott, and Andrew Toms

Original Announcement: This workshop will explore the Cuntz semigroup - an invariant of C*-algebras inspired by K-theory and recently shown to be important for classification.

In the 1970s Joachim Cuntz introduced this new invariant, based on a C*-analogue of the comparison theory that was central to work of Murray and von Neumann. It initially received some attention, but soon fell out of favor with researchers. However, in the last decade interest has been renewed. Indeed, following Toms's use of this invariant to distinguish otherwise indistinguishable algebras, a number of researchers began studying the Cuntz semigroup and a flurry of papers soon followed.

The main goal of this workshop is to clarify the role of the Cuntz semigroup in the classification program, including a discussion of related (and relevant!) problems. More precisely, the following questions, among others, will be addressed:

- How much information can be squeezed out of the Cuntz semigroup? For example, Elliott, Coward and Ivanescu have shown the Cuntz semigroup to be isomorphic to the semigroup of Hilbert modules; can this be used to prove new classification theorems? Does the Cuntz semigroup classify the "singular" cases (i.e. where the Elliott invariant is known not to be complete)?
- When is the Cuntz semigroup functorially equivalent to other invariants, such as the Elliott invariant or Thomsen's invariant? For example, can we put the recent work of Ciuperca-Elliott and Elliott-Robert-Santiago into a common framework?
- Conjecturally, strict comparison (a statement about the order structure of the Cuntz semigroup) is related to other important properties such as Z-stability or the classical notions of topological or mean dimension. Are these conjectures true?

Organizer's report:

The object of this workshop – the Cuntz semigroup – was defined over thirty years ago, but was recently proved to be isomorphic to another semigroup, one defined completely differently. As such, the first day was devoted to bringing all participants up to speed on the two pictures of the Cuntz semigroup, as well as basic techniques and results needed to work with these objects. In the morning there were two introductory lectures, one exploring the apparently different definitions and the connections between them, the other lecture outlining some important applications of Cuntz's semigroup. In the afternoon we organized three "ask an expert" groups and one "technical problems" group. In the former, an expert would field questions from participants less familiar with the group's topic, while the latter was reserved for the (relatively few) participants already on top of both definitions and all the technical results one needs to know in order to work in this area.

For the remainder of the week, afternoons were spent in working groups. Here's a summary of the main themes of discussion, as well as some of the progress made on particular problems.

(1) Answering a question of Cuntz, it was shown during the workshop that the Cuntz semigroup of a C*-algebra A is isomorphic to the Cuntz semigroup of an abelian C*-algebra if and only if A is Morita equivalent to an abelian algebra. In addition to being of independent interest, this nice result shows that some important C*-structure can be deduced from the Cuntz semigroup. Other results of this type had been known (e.g., recovering an algebra's ideal lattice from its Cuntz semigroup), but this was an unexpected addition to the known results.
(2) Partially answering another question of Cuntz, it was shown that the "open unit ball" of the Cuntz semigroup has a unique maximal element, in the simple case. Those working on this problem expressed confidence that, in general, the left kernel of a pure state defines a maximal element, however questions such as when this element is unique, remain open.
(3) Motivated by analysis of the Cuntz semigroup for so-called AH algebras, two groups made progress on the classification of these algebras in the simple real rank zero case. The first result is that unital simple AH algebras with unique tracial state, stable rank one, and infinitesimal-free K_0-group are classified up to isomorphism by their graded ordered K-theory, leading in particular to a complete description of their Cuntz semigroups. The second group replaced the tracial and K_0-group conditions with a restriction on bonding maps, and while they did not get classification, they were able to develop a previously untested line of attack—work is expected to continue beyond the conference.
(4) One fundamental question encountered when trying to use the Cuntz semigroup for classification is how to incorporate K_1-type data. (For example, there exist $A\mathbb{T}$ algebras with isomorphic Cuntz semigroups, but non-isomorphic K_1 groups.) Hence, as a first step toward a uniqueness theorem, it is natural to ask whether the Cuntz semigroup can be used to classify approximate unitary equivalence classes of unitaries. (The Cuntz semigroup is known to classify approximate unitary equivalence classes of self-adjoints, in many cases.) Though participants did not solve this problem in general, important progress was made in some nontrivial cases. For example, when the unitaries are assumed to be exponentials, or if the ambient algebra is the Jiang-Su algebra. It is hoped that this progress will lay the foundation for future collaboration, and we also note that the group working on this included both senior researchers and graduate students.
(5) One group of participants spent three days seeking the right definition of a bivariant version of Cuntz's semigroup. Though this was unsuccessful, several possibilities were definitely eliminated and it seems very likely that this work laid the groundwork for future collaboration. The group included several graduate students and it is hoped that future work will include them.
(6) A group which met three times during the week considered the question of whether elements of the Cuntz semigroup of a unital simple infinite-dimensional C*-algebra can be approximately divided by an arbitrary natural number. (A positive answer is known to have strong consequences for the classification of C*-algebras.) This is a very technically demanding and difficult question, and no major breakthroughs were achieved. There was, however, a new approach to the question, which will be tested in the coming months. It is based on the fact that the desired divisibility result holds

in the real rank zero case, and the course of the proof can at least be attempted with projections replaced with positive elements.

(7) A final group met only on Friday, with an idea to prove that in fact the answer to the question in (6) is negative. They observed that as with vector bundles, there may be homotopy obstructions to extending positive operators over a closed subset of a metric space to the whole metric space unless one allows some variation in rank. It is hoped that an analysis of the homotopy type of the space of nonnegative definite matrices with rank between k and l will shed light on the magnitude of this rank variation. If it can be shown to be on the order of the dimension of the underlying space, then there is some hope that a counterexample to (6) can be found.

To facilitate future work on the Cuntz semigroup and improve the accessibility of the subject for graduate students and young researchers, an effort was made to collect the statements of a multitude of technical results that are scattered throughout the literature. Proofs, however, could not be collected on short notice, only references. One young participant proposed a solution: he offered to create a community edited webpage, in the style of Wikipedia, where basic facts related to the Cuntz semigroup can be added – with proofs. This is a great idea and the organizers will do their best to ensure that it comes to fruition.

In summary, the meeting was very successful. Many interesting problems were communicated to a wider audience, new problems were suggested, some problems were solved, and many others saw material progress.

9. Cyclic homology and symplectic topology

November 9 to November 13, 2009
Organized by Mohammed Abouzaid, Eleny Ionel, Lenhard Ng, and Paul Seidel

Original Announcement: This workshop will be devoted to a significant ongoing development on the interface between algebra and geometry through the realization that the "closed string" invariants of symplectic and contact manifolds, obtained by counting periodic orbits,can often be thought of algebraically as versions of Hochschild homology and cyclic homology. The relevant algebraic techniques originated outside symplectic topology (for instance, see the work of Goodwillie and Jones on free loop space cohomology), but the connection has become increasingly close with the development of string topology and symplectic field theory (see recent work of Godin and Cohen on one side, and of Latschev and Cieliebak on the other).

This workshop will bring together specialists on the symplectic and algebraic sides with the primary purposes of understanding this emerging picture. We expect to discuss implications for the structure of symplectic invariants, as well as concrete geometric applications. Two specific issues ofinterest will be

- conjectures of Seidel describing symplectic homology as Hochschild homology, in the context of Lefschetz fibrations;
- recent work of Bourgeois-Ekholm-Eliashberg which constructs an exact sequence for contact homology under critical handle attachments.

Organizer's report:

This workshop was prompted by the recent preprint [2] of Bourgeois, Ekholm, and Eliashberg giving a surgery formula for contact homology and related invariants of Stein manifolds. It brought together symplectic topologist along with experts in topological field theory in order to understand (i) the connection with conjectures described in [3] by Seidel which relate the symplectic homology groups of the total space of a Lefschetz fibration to the Hochschild homology of a certain category built from an explicit collection of Lagrangian contained in a smooth fibre, as well as (ii) explore consequences for the algebraic structures on contact and symplectic homology.

Workshop Activities

The mornings of the first two days consisted of talks by Bourgeois and Ekholm outlining their preprint [2], and by Ganatra and Maydanskiy giving a dictionary relating these results to the conjectures of Seidel. In the afternoons, two groups formed, one focused on understanding the symplectic and contact homology of manifolds obtained by Legendrian surgery on S^3 or connected sums of $S^1 \times S^2$, while the other used the setting of cotangent bundles to try to understand a picture of these invariants from the point of view of topological field theory, with symplectic homology expressed as Hochschild homology.

On Wednesday morning, Oancea presented his work [1] on expressing contact homology as an S^1-equivariant version of symplectic homology, while Seidel further explained the connection with Fukaya categories of Lefschetz fibrations. In the afternoon, the topological field theory group compared the algebraic operations on cyclic homology with those expected on contact homology, another group sought to find invariants to distinguish symplectic manifolds with vanishing symplectic homology, while a third considered a generalisation of Seidel's conjectures to the case of Morse-Bott singularities.

On Thursday, Perutz explained his proof of a conjecture of Seidel computing the Floer homology of the monodromy symplectomorphism acting on the fibres of a Lefschetz fibration from knowledge of the Lagrangian Floer homology of the vanishing cycles. McLean showed how the Floer homology of the monodromy can be used to compute the symplectic homology of the total space. In the afternoon, one group continued the study of operations on contact and S^1-equivariant symplectic homology, while another explored some ideas proposed by Eliashberg for using the work of [2] to construct symplectic manifolds with various properties.

The final day started with a talk by Abouzaid on using symplectic homology to understand the Fukaya category of Liouville domains, followed by Teleman giving a general picture of topological field theories and their deformation theory. In the afternoon, one group discussed the Grothendieck group of Fukaya categories, while another discussed strategies for producing exotic symplectic structures in higher dimensions by applying the surgery formulae to Legendrian surgery on jet spaces.

In the organizers' opinion, the workshop was quite successful, with fruitful dialogue initiated between researchers in various areas (notably symplectic topology and homological algebra) on the workshop topic, and progress made in several areas of emphasis (see below).

Achievements

- In the case a single handle is attached to a subcritical manifold, we extended the results of Bourgeois-Ekholm-Eliashberg to give an equivalence of A_∞ structures between the Legendrian contact homology algebra of the Legendrian spheres before handle attachment and the wrapped Floer homology of the co-core Lagrangian disc after surgery.
- We clarified the status of the string cobracket of Chas and Sullivan as an operation which does not follow the topological field theory formalism. In particular, Seidel described a construction of such a cobracket on S^1-equivariant symplectic homology, as a secondary operation coming from the vanishing of the coproduct on the "positive" part of symplectic homology.
- Ekholm and Ng produced examples of simple Legendrian knots in $S^1 \times S^2$ which can be used to construct exotic smooth structures on R^8 and perhaps R^6.

Outlook

- Is there an extension of the Topological Field Theory formalism which produces operations analogous to the cobracket on contact homology?

- Can the approach of [2] determine the product structure on symplectic cohomology? In particular, it should agree with the product on the Hochschild cohomology of the Legendrian contact homology algebra? What about other algebraic structure on symplectic and contact homology?
- There is now a large number of constructions producing symplectic manifolds with vanishing symplectic homology. Are there any invariants distinguishing them?

1 F. Bourgeois and A. Oancea, *An exact sequence for contact- and symplectic homology,* available at `arXiv:0704.2169[math.SG].`

2 F. Bourgeois, T. Ekholm, and Y. Eliashberg, *Effect of Legendrian Surgery,* available at `arXiv:0911.0026 [math.SG].`

3 P. Seidel, *Symplectic homology as Hochschild homology,* available at `arXiv:math/0609037.`

10. Brownian motion and random matrices

December 14 to December 18, 2009

Organized by Peter Forrester, Brian Rider, and Balint Virag

Original Announcement: This workshop will be devoted to β-generalizations of the classical ensembles in random matrix theory. These are certain tridiagonal and unitary Hessenberg matrices, with an eigenvalue p.d.f. generalizing that of Gaussian Hermitian matrices and Haar distributed unitary matrices.

The β-generalizations lead to characterizations of the limiting eigenvalue distributions by way of certain stochastic differential equations involving Brownian motion. This in turn has been used to solve some previously intractable problems in random matrix theory, an example being the large distance asymptotic expansion of the spacing distributions for general β.

The time is ripe to use the stochastic characterization to tackle other problems fundamental to random matrix theory.

Universality: Do the bulk scaled eigenvalues in the β-generalized Gaussian and circular ensembles have the same distribution, and what if the Gaussian is replaced by say a quartic? Seemingly different stochastic descriptions apply in these cases, and the task is to show that they are in fact identical.

Phase transitions: Gaussian ensembles can be generalized to have Brownian motion valued entries, with one of the simplest initial conditions being to start all eigenvalues off at the origin except for one outlier. By tuning the value of the position of the outlier as a function of the size of the matrix, it is possible to get a critical regime, which is essentially the one studied in the context of spiked models. The problem here is to use the tridiagonal matrix models to study this setting, and to apply the findings to spiked models.

Integrability: Random matrix theory is a rich arena of integrability, with key probabilistic quantites known in terms of solutions of certain (non-stochastic) d.e.'s. One would like to use the s.d.e.'s, or other structures not restricted to the classical couplings, as a pathway to exact results for general β.

Organizer's report:

This workshop, sponsored by AIM and NSF, was devoted to β-generalizations of the classical ensembles in random matrix theory. Recent advances have put stochastic methods on center stage, thus explaining the workshop title 'Brownian motion and random matrices'.

One recalls that a viewpoint on classical random matrix theory, generalizing Dyson's three fold way, is that physically relevant ensembles are specified by the Hermitian part of the ten infinite families of matrix Lie algebras. By specifying a Gaussian weight, in each case the corresponding eigenvalue probability density function can be identified with the Boltzmann factor of a classical gas interacting via a repulsive pairwise logarithmic potential (log-gas). Curiously, the dimensionless inverse temperature β in this analogy is restricted to one of three values $\beta = 1,2$ or 4. This analogy, with the same restriction on β, carries over to the classical ensembles of random unitary matrices based on the ten families of symmetric spaces in correspondence with the matrix Lie algebras.

In work dating from the first half of the previous decade, explicit constructions were given of random matrix ensembles with eigenvalue probability density functions realizing the log-gas Boltzmann factors for general $\beta > 0$. In the cases of the classical Gaussian Hermitian and circular ensembles, these construction are in terms of certain tridiagonal and unitary Hessenberg matrices respectively. Alternatively it was shown that the β-ensembles could be realized by certain families of random matrices defined recursively.

In the second half of the the previous decade it was shown that the tri- (and bi-) diagonal matrices appearing in the construction could be viewed as discretizations of certain differential operators perturbed by a noise term involving Brownian motion. Similarly, by analyzing the recurrences satisfied by the characteristic polynomials associated with the tridiagonal and unitary Hessenberg matrices, stochastic differential matrices were derived for the charaterization of the number of eigenvalues in a given interval in the bulk. This in turn was used to solve some previously intractable problems in random matrix theory, an example being the large distance asymptotic expansion of the spacing distributions for general β.

The AIM workshop 'Brownian motion and random matrices' sought to build on these advances, and to tackle other problems of fundamental importance to random matrix theory. Three classes of problems were so identified.

- Universality. Do the bulk scaled eigenvalues in the β-generalized Gaussian and circular ensembles have the same distribution, and what if the Gaussian is replaced by say a quartic? Seemingly different stochastic descriptions apply in these cases, and the task is to show that they are in fact identical.
- Phase transitions. Gaussian ensembles can be generalized to have Brownian motion valued entries, with one of the simplest initial conditions being to start all eigenvalues off at the origin except for one outlier. By tuning the value of the position of the outlier as a function of N, it is possible to get a critical regime, which is essentially the one studied in the context of spiked models. The problem here is to use the tridiagonal matrix models to study this setting, and to apply the findings to spiked models.
- Integrability. Random matrix theory is a rich arena of integrability, with key probabilistic quantites known in terms of solutions of certain (non-stochastic) d.e.'s. One would like to use the s.d.e.'s, or other structures not restricted to the classical couplings, as a pathway to exact results for general β.

The workshop began with talks by two of the three conference organizers: Forrester and Virág. The talk of Forrester was structured about these three classes of problems, while the talk of Virág surveyed the developments stemming from the introduction of stochastic differential equations and related concepts into random matrix theory. Time was then spent on identifying specific problems for research. In this process the initial list was further specialized, and problems outside the list were formulated. An example of the latter was to quantify the random function implied by the bulk scaling limit of the characteristic polynomial for the β-ensembles. Four of these problems were chosen for future research during the week, and substantial progress was made on two: recurrences relating to alternative constructions of the β-ensembles, and spiked models at the hard edge.

In relation to the first, one recalls that the β-ensembles were originally formulatted in terms of tri-diagonal and Hessenberg matrices, and it was from these that the characterizations of β-ensemble states in terms of stochastic differential equations were derived. An alternative formulation of the β-ensembles is via Selberg integral theory. The problem posed was to investigate the scaling limit of the corresponding recurrences for the characteristic polynomials, which are distinct from those implied by the original constructions. These recurrences are in fact related to certain generalized eigenvalue problems, although no use was made of this feature. Instead a more direct analysis was employed. In the case of the Laguerre ensemble, the recurrence was shown to remain a recurrence in the hard edge scaling limit; for the circular Jacobi ensemble two regimes were identified — one about the spectrum singularity and the other about the point furtherest from the spectrum singularity. The latter both appear to relate to stochastic differential equations in the scaling limit. Moreover, in the course of this investigation it was noted that the spectrum singularity should be regarded as a Fisher-Hartwig singularity. The Fisher-Hartwig singularity has two parameters — one the exponent for the vanishing at the spectrum singularity, and the other controlling a discontinuity. A basic observation (due to Rains) made at the workshop is that by taking a limit of the latter the hard edge is reclaimed, so there is a transition from the two-parameter spectrum singularity to the hard. This fact seems to have escaped earlier attention.

In the original list of problems, it was proposed to use tridiagonal models and the methods of stochastic differential equations to study the scaling limit of the separation of the largest eigenvalue under a low rank perturbation. It turned out that just before the workshop Bloemendal and Virág had made substantial progress on the analogous effect for spiked Wihart matrices. This motivated the question of using these methods to study separation of the smallest eigenvalue in spiked Wishart models. This is a hard edge effect and thus complementary to the soft edge effect studied by Bloemendal and Virág.

The insight of Bloemendal-Virág was that critical spiking (in the population matrix) transformed the boundary condition in the charactering random operator from Dirichlet to "mixed". In the hard edge case, the random operator generates a diffusion process and the mixed boundary condition has a separate probabilistic meaning. Again one can write down PDE's for the distribution functions of now the spiked minimal Wishart eigenvalues. The consequences of this were considered at the workshop by Baik, Ramirez, Rider, Sutton, and Zeitouni.

During the week a further working group was formed around the topic of the random analytic function formed from the characteristic polynomial of the β ensembles in the bulk. Virág and coworkers have used the stochastic description to quantify this entity for $\beta > 2$. Attention was focussed on the case $\beta = 2$, and the approach taken was to seek to analyze the moments.

The two talks that began the workshop have already been recorded. On each subsequent day a number of other talks were given. In particular, Killip spoke on an approach to the averages of multiple characteristic polynomials in the circular β-ensembles using partial differential equations; Bloemendal on low rank perturbations of Wishart β-ensembles and their description in terms of stochastic differential equations; Borodin on determinantal point processes relating to the KPZ equation of stochastic growth; Valkó and Kritchevski on localization in a discretization of the one-dimensional Schrödinger equation in the presence of noise; Rains on the use of Selberg integral theory to construct β-ensemble; Breuer on the spectral measure associated with the β-ensembles; Krishnapur on random matrix and random function models with complex eigenvalues and zeros; and AIM director Conrey spoke on random matrix theory as it occurs in number theory.

11. Mock modular forms in combinatorics and arithmetic geometry

March 8 to March 12, 2010
Organized by Kathrin Bringmann, Ken Ono, and Sander Zwegers

Original Announcement: This workshop will focus on mock modular forms as they occur in combinatorics and arithmetic geometry and explore some other potential applications.

In recent years there has been a flurry of activity in the area of mock modular forms. Despite the recent rapid development, many open problems and further applications still need to be addressed. The primary goal of this workshop is to focus on some of the different areas of mathematics and physics that could benefit from the many recent advances in the subject.

Below is a partial list of topics to be discussed during the workshop. The participants are more than welcome to suggest related topics of their interest.

- Modularity of q-series: Although lots of examples exist of q-hypergeometric series that behave nicely under modular transformations (modular forms, mock modular forms, etc.), there is no general theory that can help you determine if a particular q-series is (mock) modular or not. We hope to get a better understanding of the relations between q-hypergeometric series and automorphic forms.
- Higher depth modular forms: The class of mock modular forms, being an extension of the class of classical modular forms, can be extended further to so called higher depth mock modular forms to obtain a graded ring. Furthermore, this seems to be the appropriate setting to understand certain theta functions associated to indefinite quadratic forms and other functions with near modular properties, like the quasi-modular forms and quasi-mock modular forms. We hope to get a better understanding of these functions, and how they show up in combinatorics and related areas.
- Applications of mock modular forms: Recent works have shown that mock modular forms appear as a type of generating function for derivatives of twisted modular L-functions. Mock modular forms have also appeared in connection with the representation theory of affine Lie superalgebras. We shall work to obtain a more precise relationship between derivatives of elliptic curve L-functions and the coefficients of mock modular forms, and we shall work to compute the asymptotic dimensions of characters of Kac and Wakimoto.

Organizer's report:

The primary goal of this workshop was to focus on some of the different areas of mathematics and physics that could benefit from the many recent advances in the area of mock modular forms. In the mornings there were one or two introductory lectures. On the first day, George Andrews gave an talk on various techniques and results in the theory of q-series. Don Zagier brought the participants up to speed on Nahm's conjecture, which gives a (deep) relation between the modularity of q-series and the Bloch group. On day two Wladimir Pribitkin gave an historic account of modular integrals, first introduced by Niebur in the late sixties, and Jan Bruinier explained the connection between mock theta functions and Borcherds type products. On Wednesday Sander Zwegers gave a talk on higher depth mock modular forms, a further generalization of the "classical" mock theta functions. On the last two days we saw some further applications of mock theta functions: Andreas Malmendier explained the connection between mock theta functions and the classification of manifolds, and Karl Mahlburg spoke about bounds on metastability thresholds and probabilities for generalized bootstrap percolation.

In the afternoons the participants broke up into discussion/working groups. Some of these groups were devoted to the subject of the morning talk, where the speaker would provide further details and the participants could ask more questions. We now give a summary of some of the main themes of discussion.

Nahm's conjecture: Nahm's conjecture involves the problem of understanding the overlap between the classes of q-hypergeometric functions and modular forms. The (conjectural) answer involves dilogarithms and the Bloch group. After Don Zagier gave an excellent introduction into the subject, several afternoons were spent on further discussions. These discussions were mainly brainstorming sessions where ideas were being thrown around. One possible way to attack the problem is to consider the asymptotic expansion of the q-hypergeometric function as q approaches a root of unity. In principle this technique could very well gave a proof of the conjecture in one direction: if a q-hypergeometric series is modular, then the corresponding element in the Bloch group is torsion. This approach has been worked out for the rank 1 case, but for larger values of r it becomes computationally very hard. Several ideas were discussed to get better asymptotics and how to make better use of them. In the other direction we have the problem of finding an actual identity once we know that a particular q-hypergeometric series is modular. Several techniques coming from the theory of q-series were discussed and also how they "translate" on the Bloch group side. While no actual results were obtained, several new ideas were given and this will be the start of more future research.

Algebraicity: Although there are natural algebraic structures within the classical theory of modular forms (e.g. singular moduli, Hecke eigenvalues, Fourier coefficients of newforms, etc.), we have come to understand that the situation is more complicated in the theory of harmonic Maass forms. In this regard, there were daily discussions on the algebraicity of mock modular forms of integer weight and half-integral weight, as well as discussions on the algebraicity of CM values of suitable harmonic Maass forms. In the context of harmonic Maass forms with integer weight, it is known (by work of Bruinier, Ono, and Rhoades), that mock modular forms associated to CM forms have algebraic coefficients. Luca Candelori gave a lecture on his thesis, where (in the case of weight 2) he refined this result to show that the mock modular forms are defined over the minimal possible field of definition. There were discussions on how to generalize this result to all integer weights. For half-integral weights, Bruinier spoke on a recent joint paper with Ono which equates the vanishing of derivatives of modular L-functions to the algebraicity of corresponding Fourier coefficients. Many discussions involved the problem of making this phenomenon more precise. Namely, obtain a direct formula for these algebraic coefficients in terms of the derivative of the L-function, or the heights of Heegner points, or periods. There was also many discussions on the algebraicity of harmonic Maass forms at CM points. Preliminary results were obtained in this direction.

Percolation: By work of Andrews, Holroyd, Liggett and Romik it turns out that mock theta functions are related to certain cellular automata models. From an automorphic point this corresponds to understanding products of Ramanujan's mock theta functions with classical weakly holomorphic forms. Karl Mahlburg gave an introduction in to the subject and described how different areas are connected here: number theory, combinatorics, probability, and cellular automata models. In the discussion following, particular asymptotic formulas and how the interplay between the areas might help here, were discussed.

In particular the question arose whether certain injections for partitions may help to understand certain probability events. This is still an ongoing discussion.

Over all, the workshop was very successful. Various applications and new ideas were communicated to a wider audience and progress was made in several directions.

12. Localization techniques in equivariant cohomology

March 15 to March 19, 2010

Organized by William Fulton, Rebecca Goldin, and Julianna Tymoczko

Original Announcement: This workshop will be devoted to localization techniques in equivariant cohomology. Localization techniques in equivariant cohomology are a powerful tool in computational algebraic topology in the context of a topological space with the action of a Lie group. The development of these techniques has led to an explosion of research, including groundbreaking work in many different fields, especially algebraic combinatorics, algebraic geometry, symplectic geometry, and algebraic topology.

The goal of this conference is to relate the geometric and combinatorial aspects of what has been dubbed "GKM" theory by bringing researchers from combinatorics and geometry/topology together, so that workers in each of these fields can benefit from the insights and experience of those in the other. The conference will address open questions such as the following:

- Can the combinatorial techniques of GKM theory be extended to a wider class of spaces, even if the image of the inclusion map cannot be described? These families could include singular spaces, infinite dimensional spaces, stacky spaces (including Deligne-Mumford stacks), and spaces with specific kinds of group actions or geometric structures.
- For what larger class of manifolds than G/P does the Kostant program extend? Resolving this would entail finding geometric reasons behind the combinatorial rules in G/P. What specific kinds of group actions or geometric structure would be required? Answering this question would allow us to apply these powerful combinatorial techniques to reveal geometric and algebraic structure (such as intersections and cohomology rings of all types) in a much broader context.
- Can the techniques of GKM theory be used to prove combinatorially positive results in Schubert calculus or its generalization in an appropriate geometric setting? A combinatorially positive result interprets structure constants in the cohomology ring as the cardinality of a set; an answer to this question might generalize combinatorial objects such as puzzles, cartons, or tableaux to an algorithm that counts paths within graphs, and give closed positive formulas for structure constants in an equivariant cohomology ring.

This conference will gather researchers from each of the main approaches to equivariant cohomology: algebraic geometry, symplectic geometry, algebraic topology, and algebraic combinatorics.

Organizer's report:

The conference attendees consisted of mathematicians from different areas of mathematics who work on localization techniques in equivariant cohomology. The fields represented at the conference included algebraic topology, symplectic geometry, combinatorics, and algebraic geometry; all of these fields have contributed significantly to the study of equivariant cohomology, but often without understanding of related work in other fields. Our goal was to open communication between mathematicians in different fields, to initiate deeper mathematical work and prompt collaborations across disciplines. As such, an important goal was to establish a common vocabulary about the definitions, techniques, and even motivating questions that different researchers considered. (Unofficially, we tried to avoid research talks in favor of surveys.)

The first morning had two survey talks. Sara Billey described what localization meant from her perspective as a combinatorist; she focused on the existence and explicit construction of computationally-useful bases, particularly for topological spaces with important combinatorial attributes (like flag varieties, Grassmannians, and Schubert varieties). Megumi Harada gave a symplectic geometer's perspective on the same, introducing symplectic manifolds M with Hamiltonian T-actions, together with the inclusion of fixed points $\iota : M^T \hookrightarrow M$ and the map ι^* that inclusion induces on T-equivariant cohomology. Her talk addressed some conditions under which the induced map $\iota^* : H_T^*(M) \to H_T^*(M^T)$ is injective. These conditions—and the general notion of *equivariant formality*, which has substantively different meaning for different kinds of mathematicians, but is related to the condition that $H_T^*(M)$ be a free module over $!H_T^*(\mathrm{pt})$—became an ongoing theme in the conference.

The first afternoon had a group discussion with several mini-talks. Volker Puppe and Matthias Franz fleshed out ideas about how algebraic localization could be used to identify the image of the map ι^*. Megumi Harada continued her talk with a discussion of ABBV localization (a theorem of Atiyah-Bott-Berligne-Vergne describing how to compute equivariant integrals); Allen Knutson also made some short comments on the same theme. The day ended with a group discussion of open questions.

The second morning had two survey talks. The first, by Tom Braden, described from an algebraic topologist/geometer's perspective the basic constructions that are sometimes called GKM theory. His fundamental object was a complex algebraic variety X with a T-action such that the number of T-fixed points is finite, the number of one-dimensional T-orbits is finite, and X satisfies equivariant formality (especially that $H_T^*(X)$ be free over $H_T^*(\mathrm{pt})$). In this context, he described explicit combinatorial calculations that give the ring $H_T^*(X)$ based on a graph (called the GKM graph, the moment graph, or the labeled one-skeleton) that encodes the fixed points and one-dimensional T-orbits of X. Sue Tolman gave the second talk, and described a symplectic geometer/algebraic topologist's approach to similar questions. She discussed the extent to which the moment graph of a symplectic manifold M determines the manifold, and how to use the graph to get formlas for computing (via computationally convenient bases).

On the second afternoon, we broke into three groups. Tom Braden gave the largest group an introduction to intersection homology, beginning with the general constructions and theory of intersection homology and ending with the analogy of GKM theory for intersection homology. A smaller group discussed specific calculations in GKM theory, doing some examples outside of Lie type A and asking (and then answering) what the natural map $G(k_1, n_1) \times G(k_2, n_2) \to G(k_1 + k_2, n_1 + n_2)$ induced on equivariant cohomology is. The third group worked on the topic *Beyond equivariant formality* and generated provocative (research-level) ideas about how to proceed with GKM-like analysis without the condition of equivariant formality.

The third day began with two algebraic geometers (Allen Knutson and Dave Anderson) describing toric degenerations: what they are (geometrically and algebraically) and how they can be used, particularly in Schubert calculus contexts (namely, to answer questions about the cohomology of a flag variety or Grassmannian with respect to the basis of Schubert classes). The afternoon saw a vigorous introduction to K-theory, with several tag-team speakers and active audience participation (and chalk-grabbing). Speakers discussed the essential differences between T-equivariant cohomology and T-equivariant K-theory (where the former is a module over a polynomial ring, the latter is a module over Laurent polynomials). They also described and gave examples of the analogue of GKM theory for equivariant K-theory.

On the fourth day we turned to Schubert calculus, which studies the cohomology ring (for many different cohomology theories) of a Grassmannian (or other partial flag variety) in terms of a natural geometric basis of *Schubert classes*. Alex Yong gave a combinatorial perspective on Schubert calculus. He described how Schubert polynomials are a combinatorial object that mediates between classical questions in Schubert calculus and geometric questions about the singularities of Schubert varieties. Linda Chen then spoke from an algebraic geometer's perspective, and presented the three main goals of Schubert calculus from her point of view: 1) an explicit presentation of the cohomology ring that is preferably both algebraically and geometrically natural in some sense; 2) the *Giambelli problem* (to write down an arbitrary Schubert class in terms of a nice additive module basis of special Schubert classes); and 3) the *Pieri problem* (to understand the product of a special Schubert class with an arbitrary Schubert class).

In the afternoon, the group generated a table of what the participants know about Schubert-calculus. (This table can be found on the wiki page associated with this conference.) We then broke into groups, the largest of which talked about affine Schubert calculus (how it relates to classical Schubert calculus, and what's known and unknown). Two similarly-small-sized groups focused on more specific questions: finding equivariant Pieri rules and finding new families of topological spaces that satisfy GKM conditions, but are not homogeneous spaces.

On the morning of the fifth day, we had three shorter talks. Bill Graham spoke about joint work with Sam Evens on the Belkale-Kumar cup product. Hugh Thomas spoke about combinatorial Littlewood-Richardson rules via jeu-de-taquin, a fundamental operation in the study of Young tableaux. Matthias Franz spoke about equivariant formality: what different mathematicians might mean by it (except rational homotopy theorists, who mean something completely different). After listing several related conditions that could be referred to as equivariant formality, he then described which are equivalent over field coefficients and which imply others. He finished by discussing what happens with integer coefficients. The morning was long, so the afternoon was short, mostly consisting of smaller conversations and some discussion of open problems.

Outcomes: As of this writing, several new connections and collaborations have been formed. Existing collaborations were energized; some started new work. At least one group of coauthors revised a paper in the light of presentations from mathematicians outside their field, which made them realize that their existing results were in fact stronger and more far-reaching than they had realized. One participant said it was the highlight of graduate school (so far); a very experienced senior faculty member said it was the best conference he had ever attended.

More concretely, conference attendees generated a list of open problems and an annotated list of references. The attendees also generated an annotated table of what's known and not known (to conference participants) in Schubert calculus. The references, open problems, and table are available on the conference wiki; we hope they will have a longer life as a larger public edits, corrects, and updates them.

13. Computational optimization for tensor decompositions

March 29 to April 2, 2010

Organized by Rasmus Bro, Michael Friedlander, Tamara G. Kolda, and Stephen Wright

Original Announcement: This workshop will be devoted to facilitating the development of new decomposition methods and to provide fundamentally new insights into both tensor decompositions and numerical optimization. During the past decade, there has been an explosion of interest in tensor decompositions as an important mathematical tool in fields such as psychometrics, chemometrics, signal processing, numerical linear algebra, computer vision, numerical analysis, data mining, neuroscience, and graph analysis. Tensor decompositions are generalizations of matrix decompositions (which have proved to be a vital tool in many areas of science and engineering) to N-way tensors, where N is greater than 2. In many circumstances, N-way tensor representations allow a much more natural framework for representing relationships between elements in a data set than do traditional matrix representations. Tensor decompositions have the potential to revolutionize our scientific capabilities in such applications as environmental monitoring, medical diagnostics, cybersecurity, anti-doping testing, telecommunications, and more.

Further advances in tensor decompositions depend critically on advanced optimization algorithms. Computational tools have not changed significantly in the past four decades and are often based on a simple alternating least squares (ALS) approach, which is often slow and comes with no guarantees of convergence to a useful solution. Despite these drawbacks, it remains the method of choice because it is quite general and because it allows for useful modifications (for example, to problems with missing data). Recent work has shown that optimization methods other than ALS can provide superior solutions in specific situations. Our workshop seeks to further this line of research by bringing together leading experts in numerical optimization and tensor decompositions, with the purpose of developing optimization-based tensor decomposition methods that are robust, accurate, numerically stable, and scablable. Furthermore, these methods should be able to allow constraints such as nonnegativity to be imposed on the parameters; they should allow missing data to be efficiently and accurately handled; they should enable sparse data and sparse solutions; and they should be able to handle formulations that involve alternative loss functions such as (generalized) weighted least squares.

The goal of this workshop is to foster a new scientific community to facilitate the development of new decomposition methods and to provide fundamentally new insights into both tensor decompositions and numerical optimization. This community is expected have an impact in many diverse areas (including those listed above) in the years to come.

Organizer's report:

In the past decade, there has been an explosion of interest in tensor decompositions as an important mathematical tool in fields such as psychometrics, chemometrics, signal processing, numerical linear algebra, computer vision, numerical analysis, data mining, neuroscience, and graph analysis.

Further advances depend critically on algorithms that are robust, accurate, numerically stable, and fast. Despite widespread interest, the computational tools available to practitioners have not changed dramatically for nearly four decades. State-of-the-art methods are based on simple alternating least squares (ALS); this approach is often slow to converge and there are few guarantees that it will converge to a useful solution. Some researchers have observed that more sophisticated optimization methods provide superior solutions and have presented preliminary computational results to support their claims.

The AIM workshop on tensor decompositions aimed at generating a fruitful collaboration between the tensor algorithm researchers and the optimization community. Thanks to a schedule of full-group talks and demonstrations, focused breakout groups, and a very engaged set of participants, the workshop turned out to be an inspiring and constructive meeting providing many new insights and starting many collaborations that we believe will help accelerate the pace of developments in this field.

Our focus in this workshop was more on the algorithmic and computational aspects of tensor decompositions than on the very rich theory associated with the area, though the participants included several people who are experts in the latter area.

Some of the most important issues that arose in the workshop are described below.

The Alternating Least Squares (ALS) Algorithm for fitting CANDECOMP/PARAFAC

In ALS, each factor in the tensor product are optimized in turn (while the others are held fixed) in order to best match the target tensor according to an element-wise sum-of- squares criterion. Thus, each subproblem is a linear least-squares problem, but one with unusual properties: it is typically highly overdetermined, and the cost of computing the right-hand side in the normal-equations formulation is typically much higher than the cost of forming the coefficient matrix. ALS is attractive because it exploits an obvious structure in current formulations; however, it can be slow to converge and unstable in many situations. Despite these obvious problems, ALS has remained the main underlying engine since the beginning of the field. We anticipate that it will be difficult to completely replace ALS at least in the foreseeable future. Hence, at the workshop, there were several groups working on improving on this basic method and on improving our understanding of its behavior.

Speeding up ALS by Sampling

The most expensive operation in each ALS subproblem is computation of the right-hand side of the normal equations. One group at the workshop tried to reduce the cost of this operation by random sampling. In computing the matrix-vector product to form the right-hand side, only a subset of rows of the matrix are used. Different sampling schemes were tried (by modifying the Tensor Toolbox code) and the approach was found to be promising, with speedups of about 4 on a number of random problems of various dimensions. This appears to be a promising direction for future research. Another notable aspect of the sampling approach is that the noise it produces in the gradient may help in avoiding local minima (see below).

A somewhat different approach, also involving sampling and randomness, is to use stochastic gradient methods on the full problem (not in the ALS setting). Such methods have proved popular recently in other settings, for example machine learning and stochastic optimization, but their use here would require extension to the nonconvex case. There was not time during the workshop to investigate the approach, but we view it as a promising topic for follow-up work.

Convergence Behavior of ALS

One group tackled the issue of understanding convergence of ALS. Apart from the broader issue of local minima (discussed below), what can be said about the conditions under which ALS converges, and what is its rate of convergence under these assumptions? What is the connection to solving systems of equations? ALS works most of the time" in practice but can we find a better characterization of when this is true, for example conditions analogous to those that ensure that compressed-sensing recovery works? ALS iterates certainly converge to a limiting function value, because the sequence of function values is lower bounded by zero and it is monotonically decreasing. However, iterates often continue to change in large steps. Even worse, the accumulation points in some cases do not appear even to be local minima; rather they iterates appear to be stuck in a cycle. How should poor conditioning and ill conditioning of the least-squares subproblems be handled? What is the connection between the conditioning of the subproblems and poor convergence? What properties of the tensor lead to ill-conditioned subproblems? Can a more judicious choice of rank (i.e., number of columns in each matrix factor) improve performance of ALS, and how can a good rank be identified?

One line of analysis follows by recognizing that ALS is a block coordinate relaxation algorithm (i.e., a Gauss-Seidel method). The connection provides no panacea, but gives us a starting point.

Several possible angles of analysis were discussed during the workshop, particularly on the last day. There have been follow-up investigations after the workshop, but this area remains work in progress.

Modifications of ALS

One approach suggested for modifying ALS was the use of more elaborate line-search approaches. ALS traditionally steps to the minimum of each least-squares problem a greedy approach. However, there may be some benefit to over- or under-relaxing the steps in some systematic way. (Such techniques have proved to be useful in other contexts, e.g., solving sparse linear equations.) Another idea is to combine search directions in each of the factors and take a combined step. A minimizer along the combined search direction could be found by solving a higher-order scalar polynomial. Again, such approaches have proved useful in other optimization contexts. Other possible modifications include the use of damping (i.e., use of a Levenberg-Marquardt parameter) to improve the conditioning of the least-squares subproblems, which are often rank deficient.

Time did not permit serious progress to be made at the workshop on these issues, but we hope they can be addressed in the follow-up.

Local Minima

Being a nonconvex problem, the tensor decomposition problem may yield solutions that only locally minimize the misfit between the target tensor and the factors. Can we say something about the number of local minima? How deep, typically, are the wells containing local minima? If they are deep (as happens in other examples, e.g., energy potentials for protein folding) then it is hard to design algorithms that avoid them.

Can we incorporate techniques into the algorithms that make convergence to non-global minima less likely? For example, accepting some steps that increase the objective (as in simulated annealing) or using non-monotone techniques rather than the standard descent techniques are possibilities.

Some time was devoted to this topic during the meeting but not enough it remains an interesting topic for further work.

General Optimization Techniques

One group asked how much progress could be made by treating the tensor factorization problem as a general nonlinear optimization problem. They thus modeled the problem (and its nonnegative variants) by using AMPL and submitted problems to the NEOS Server for solution, using standard optimization software. This approach provides a baseline for performance of specialized methods. Preliminary indications were that the approach is successful but that algorithms that exploit the structure (which were the main focus of the workshop) are likely to me more efficient, particularly on large problems. The group was able to use the modeling power of AMPL to handle difficult problems without explicitly reformulating them. However, this is just a demonstration of the potential power of using more sophisticated optimization techniques. More work is needed to specialize the useful techniques for tensor model fitting.

Alternative Loss Functions

Most tensor decompositions are designed to minimize sum-squared error: implicit in this formulation is a Gaussian noise model for the observations. Other loss functions could be considered. If they are convex, then the subproblems arising in the alternating directions approach will also be convex (as in ALS, for the sum-of-squares loss function). At the workshop, new algorithms based on L1 fitting were developed and it was shown in practice that such algorithms produced results that were essentially unaffected by shot- noise and spurious data elements. In this sense they were robust, and contrasted quite dramatically with the sum-of-squares loss function which produces results that are seriously distorted by observation errors of this type.

Constraints in Compressed Representations

The technique of compression identifying and working in primary subspaces of a tensor, with smaller total dimension has been known and used for several decades, and is helpful in reducing computation time. What is less well understood is how to impose structure and constraints in the compressed representation. Specifically, it was now known how to impose the requirement of nonnegativity of the factors in the compressed setting. (As a related issue, PARAFAC2 has a hidden mode, but it has not been shown whether it is possible to impose constraints in that mode.) At the workshop, the use of AMPL (mentioned above) showed that showed that it is indeed possible to build nonnegativity into the compressed structure. Further work is needed to make the suggested approach practical but a proof-of-principle was obtained.

Testing new solutions

How should we go about creating data? The field desperately needs real datasets to test new methods on. The data may be from real-world problems or artificially generated. We need different types of problems as well those that are known to have troublesome local minima, degeneracies, etc. We also need large problems that are so big they dont fit on a standard computer. We need both dense and sparse problems. We are motivated by other communities that have compiled useful data sets. (The netlib set for LP and the SPARCO set for compressed sensing come to mind.) For our goal of engaging the optimization community on tensor problems, such data sets would allow all papers to use the same data and comparisons. It is also useful to understand the data by having a detailed explanation of what the data actually is, what it means, etc. To this end, it was decided to make a more elaborate web-site containing data sets with known problems and importantly with known solutions. Rasmus Bro will start this up.

Tensor completion

We considered the tensor analogue to matrix completion how to fill in missing values in your data when it is known to have underlying low-rank structure or something similar (e.g., low-rank plus noise, etc.). We can always matricize a tensor and use matrix completion techniques, but this approach ignores the multilinear structure and may be more expensive and less accurate.

One group also considered the theory of tensor completion. In the matrix case, we can say a lot about the nuclear norm of a matrix. The group considered extensions of the matrix nuclear norm in the tensor case. There are two variations. One is a theoretical construct that considers the minimum sum of weights of a rank-one factorization. The other is to average the nuclear norms of the tensor unfolded along each mode.

Many other subjects were proposed during the workshop but not discussed in detail due to lack of time and hectic activity in the above subjects. The most important of these are mentioned below. We mention in particular some of the questions that remain to be addressed in each category.

Manifold of Solutions

Due to permutation and scaling ambiguities, any CANDECOMP/PARAFAC tensor decomposition actually defines a manifold of possible solutions. A similar situation holds for the Tucker decomposition. This raises several questions: How we can isolate solutions on the manifold? What parametrizations of the manifold are useful?. Which of these parametrizations leads to each local minimum being a unique subspace? It is possible to remove the scaling by introducing constraints or regularizing the problem. In these cases, how are the manifolds represented in a way that's tractable? You have an equivalence class, as long as you have a point in the set. Is the set of solutions a manifold, or a set?

Rank

It is generally difficult to determine the numerical rank of a data tensor. It is well known that the determination of rank is NP hard and that low-rank approximations may not exist. What are practical approaches for determining the numerical rank of a tensor? The current approach is just to try different ranks until one has a good fit without over-fitting (an art more than a science). One idea that has been seen in the literature and was explored at the workshop was the idea of cross-validationi.e., leaving out some data and seeing how well the model fits the left out data to determine the rank.

Are there examples/classes of problems where we cannot compute the rank? What are the properties of such examples? For example, we might fail to fit a rank-5 tensor, but this does not assure that no rank-5 approximation exists the issue may be that the optimization routine simply failed to find it.

Convex relaxation

Can we use convex relaxations (or other norms) to find low-rank tensors? In the matrix case, we can relax the rank to a convex problem, using the nuclear norm. Can we similarly relax the tensor rank?

Sparsity and missing values

How do we handle sparse, missing value, nonnegative constrained problems? How do we do this faster than existing methods? How do we incorporate knowledge of problems into development of new (better) methods? How do we incorporate knowledge of problems into development of new (better) methods?

Damping

How do we use regularization in the optimization problem? Does regularization keep you away from bad local minima? A damping term, such as that used by the Levenberg- Marquardt method, seems to generally speed up convergence, but how does it affect the solutions obtained?

How do we leverage problem characteristics in computing and/or choosing models?

Can we check/verify that multiway structure actually exists in the nerlying application, so that the tensor framework is appropriate? Sometimes you hit the jackpot, e.g. in the case of fluorescence, there is clear physical support for multiway analysis. In general, however, how do we know that the data contains such a structure? Physical understanding may be be lacking. In the case of social data, all bets are off! There should be rules of thumb for knowing if you are doing a good job; some of this appears in a paper by Richard Harshmann called How can I know its real?"

What kind of structure is of interest in the solution?

How can this be addressed in the objective/constraints? A number of people mentioned sparsity. Can sparsity be imposed on the factors? There is some previous work by Morten Mrup on sparse tucker, using a component-wise L1 norm to enforce sparsity, but more work needs to be done.

Conclusion

The goal of the proposed workshop was to discover novel optimization approaches for solving tensor decomposition problems and to spark new lines of research in this area by bringing tensor experts together with optimization specialists. We believe it was successful. The sponsored participants remained engaged and focused during the week. Unsolicited participants (such as Ben Recht, who spoke on the second day) enlivened the proceedings and contributed significantly to the success of the event. Connections were made and strengthened. We look forward to seeing how well the directions identified at the workshop pan out in the months and years ahead.

The organizers are most grateful to AIM for their support and involvement in the workshop at their excellent facility. Their well-grounded ideas for workshop structure were familiar to only a few participants; for others, the group-focused structure was an adventure, but an exciting one. We believe that the ultimate productivity of the workshop will be much higher as a result of this intensive, hands-on approach.

14. Supercharacters and combinatorial Hopf algebras

May 17 to May 21, 2010

Organized by Nantel Bergeron, Persi Diaconis, Jean-Yves Thibon, and Nathaniel Thiem

Original Announcement: This workshop will bring together experts in the emerging area of supercharacter theory and experts in Hopf algebras. Super characters/classes are certain unions of irreducible characters/conjugacy classes which fit together to give a rich, new combinatorial structure that allows analysis for previously intractable problems such as the the group of upper triangular matrices with coefficients in a finite field Hopf algebras have given a useful framework for studying the representation theory of towers of algebras, following the classical relationship between the representation theory of the symmetric group and the ring of symmetric functions. Recent work has suggested that similar noncommutative Hopf algebraic structures might exist for super-representation theories. In particular, this workshop will investigate the case of finite groups of unipotent upper-triangular matrices.

Organizer's report:

The primary goal of the this workshop was to establish a line of communication between researchers in the emerging area of supercharacter theory and experts in combinatorial Hopf algebras. We decided to focus on the following problem which arose from the representation theory of the group of upper triangular matrices with entries in a finite field. In the theory of "super characters and super classes" the natural objects are all indexed by set partitions. There is another actively studied object with a basis indexed by set partitions, the symmetric functions in non-commuting variables. Several of us felt that there might be a connection between these two subjects. This became:

Problem. *Is there a Hopf algebraic connection between the Hopf algebra of symmetric functions in non-commuting variables and the ring of supercharacters of the unipotent upper-triangular matrix groups over a finite field with two elements?*

At the time of the workshop, it was known that there were many algebra isomorphisms between these two spaces, but it was very unclear whether a corresponding Hopf structure isomorphism was possible.

Our workshop not only succeeded in discovering an explicit isomorphism between the two spaces, but also in tackling the following variations:

- A Hopf algebra isomorphism between the ring of supercharacters of the unipotent upper-triangular matrix groups over an arbitrary finite field and a Hopf algebra of symmetric functions in colored non-commuting variables.
- A symmetric function realization for the Hopf algebra dual space of the ring of supercharacters of the unipotent upper-triangular matrix groups over an arbitrary finite field.

The participants have now written a preprint "Supercharacters, symmetric functions in noncommuting variables, and related Hopf algebras" (arXiv:1009.4134v1), detailing the results of the workshop.

Workshop details

We began the workshop with two survey talks: Nat Thiem gave an introduction to supercharacters with an emphasis on the case of unipotent upper-triangular matrices, and Nantel Bergeron gave an introduction to combinatorial Hopf algebras with an emphasis on the Hopf algebra of symmetric functions in non-commuting variables. That afternoon was spent in discussion groups hoping to address confusions from the morning talks in greater detail. It also marked the start of project, spearheaded by Franco Saliola, of implementing the Hopf relations into SAGE.

The second morning's talks gave some additional background material: Mike Zabrocki described the classical connection between the characters of the symmetric group and the Hopf algebra of symmetric functions (which was to serve as a model for our problem), and Marcelo Aguiar broadened the picture to the slightly more general setting of Hopf monoids. The afternoon was dedicated to actually begin understanding the combinatorial underpinnings of the two structures we were studying, and – to our surprise – we found an answer to our problem in the course of these discussions.

With this success under our belt, we decided to broaden our scope. To facilitate this goal, we had three more general talks on the third morning: Jean-Yves Thibon gave a more general talk about combinatorial Hopf algebras, Carlos André described how some of the nice features of the supercharacter theory of our main example generalize to other settings, and Anders Henrickson discussed some ways to construct new supercharacter theories from existing ones. As our isomorphism from the previous day had raised more questions than it had answered, we spent the afternoon in groups tackling some of these different questions. This afternoon we managed to not only find the above-mentioned variations to our problem, but we also found some more negative results, such as the fact that our supercharacters were not Schur positive under this isomorphism.

On the fourth day, Persi Diaconis began with a talk to address a concern that had been nagging many of the participants throughout the workshop: It is well-known that the usual character theory of the unipotent upper-triangular matrices over a finite field is wild, but most participants only had a vague notion of what "wild" means mathematically. With this clarified, Kay Magaard gave a talk about work with Tung Le on decomposing supercharacters into more irreducible characters. Nantel Bergeron finished off the morning with a talk clarifying the meaning of the antipode map. The afternoon was spent beginning to think about problems that would not be realistically solved in the course of the workshop. Some problems that were considered included: what are other nice supercharacter theories for the group of unipotent upper-triangular matrices, and can one glue together supercharacter theories of the symmetric group to get a combinatorial Hopf algebra related to symmetric functions in some way?

The final day of the workshop was largely focused on establishing what we had done and looking to the future. Eric Marberg described other towers of groups which have a combinatorial supercharacter theory description, and Franco Saliola gave an introduction to the use and procurement of the new SAGE package. Then the focus turned towards planning next moves. We decided to write a joint paper among the participants cataloguing our results, and in the afternoon we listed all the open problems we had discovered over the course of the workshop.

Outcomes

We had a wide range of participants, from first year grad students to senior faculty, and all seemed to find some niche to actively contribute to the overall project. Participants coming from either side of the initial divide (supercharacter and Hopf algebras) learned a great deal about both subject areas. Our biggest contribution seems to be the idea that one can give a representation theoretic underpinning to a combinatorial Hopf algebra by considering supercharacter theories in addition to the usual representation theories.

Tangible products developed by the workshop include the following.

- A workshop preprint which is submitted for publication.
- An implementation of the supercharacter theory of unipotent upper-triangular matrices over a finite field into SAGE, now available in the SAGE combinat package.
- An implementation of the supercharacter theory of unipotent upper-triangular matrices over a finite field into muPad.
- A list of open problems.

15. Low dimensional structures in dynamical systems with variable time lags

June 7 to June 11, 2010

Organized by John Mallet-Paret, Roger Nussbaum, and Hans-Otto Walther

Original Announcement: This workshop will focus on low (Hausdorff) dimensional structures in differential delay equations with state dependent time lags. The study of such structures seems to offer the best hope of progress in the broader field. We mention some examples below.

Periodic solutions of differential delay equations often play a central role in understanding the dynamics of the equations, but even proving existence of such solutions may be nontrivial. Related problems include uniqueness and stability questions, regularity (real analyticity) of solutions, and limiting shapes of periodic solutions under singular limits. Nontrivial examples are already provided by equations such as $ax'(t) = f(x(t), x(t-r))$, where $a > 0$ and $r := r(x(t))$, and f and r are given functions.

The local theory of invariant manifolds is reasonably well-developed for differential delay equations with constant time lags, but extending the theory to the state dependent case presents difficulties, e.g., in proving higher order differentiability of invariant manifolds. Related problems are also present for compact attractors, which are typically known to have finite Hausdorff dimension in the constant time lag case, and suspected so for the variable time lag case.

The presence of a Morse decomposition for the maximal attractor occurs for certain classes of systems with a single delay, or a cyclic structure, with a signed feedback. Many such systems, under generic conditions, are of Morse-Smale type and much effort has gone into studying their global structure. Broadening these results to systems with multiple delays is a major open question of much significance, both for the theory and scientific applications.

One goal of the workshop will be to bring together established figures in the field with younger researchers who may become the next generation of leaders. Since many of the equations which have been extensively studied have their origin in simplified models from applications, another objective is to facilitate conversations among researchers with widely varying degrees of interest in applications. We expect significant and fruitful advances in the field to emerge from these interactions.

Organizer's report:

The workshop discussed differential equations with time lags which are state-dependent. Such equations are not covered by the standard theory of functional differential equations, as it is presented in monographs which appeared during the past few decades: see [2, 1]. The reason for this deficiency arises from a lack of smoothness of the delay as a function of the state and its history. Motivation to study differential equations with state-dependent time lags comes from numerous models in science and technology. While it is often agreed that delays when present in a dynamical system are state-dependent, the specific form of state-dependence remains hard to describe in many cases.

We had 21 participants (18 invitees and 3 applicants). They came from North America (USA 8, Canada 3), Latin America (Chile 1), and Europe (EU 8, Switzerland 1), among them 2 graduate students and 4 postdocs. There were experts in the mathematical analysis of differential equations with state-dependent delay, newcomers (about a third of the participants), and experts in numerical analysis and modelling.

Each morning started with 2 introductory lectures, followed in the afternoon by shorter presentations and group discussions, which became more focussed and intense in the course of the week. Friday afternoon we collected a list of open problems which we believe to be crucial for further progress.

Monday began with an introduction by Walther into existence theory for the initial value problem on a Banach manifold which is defined by the equation considered. This approach yields a semiflow of differentiable solution operators, and tools from dynamical systems theory become applicable. Also addressed were the concept of deferred action, which motivates the study of algebraic-delay systems with the delay given by an algebraic equation, the phenomenon of reactions in reversed temporal order, whose mathematical description relies on *variable* time lags, and Poisson's first paper on differential equations with state-dependent time lags, from 1806 [8].

The second lecture, by Krisztin, introduced the construction of continuously differentiable local invariant manifolds. In the case of differential equations with state-dependent delay this is considerably more involved than for other types of differential equations. Higher order differentiability of the finite-dimensional local center and unstable invariant manifolds requires a set of hypotheses which is specific for state-dependent delay. For stable and other *infinite-dimensional* invariant manifolds higher order differentiability is unknown.

The afternoon session consisted of several shorter contributions, embedded in a discussion in which we tried to identify themes for informal working groups.

Mallet-Paret started with a brief introduction to the results [4, 5, 6] obtained jointly with Nussbaum concerning the asymptotic shape, uniqueness and stability of periodic solutions for a prototypic class of nonlinear autonomous equations with a small parameter. Here the asymptotic shape of a periodic solution is a set in the plane, and an important tool to find it is the *backdating map* in dimension 2. Mallet-Paret then discussed related open problems, concerning the form of the delay and the possibility of extensions to equations with several delays.

Barbarossa described her project on population models, which include equations of neutral type and unbounded delay. For neutral equations with state-dependent delay the basic theory along the lines of the first lecture is more challenging and thus less developed, with smoothness results weaker. In particular, linearization at equilibria is still an issue, and existence of local invariant manifolds is an open problem.

Stumpf presented his recent work about an equation modelling a price. He established existence of a 2-dimensional global center-unstable manifold with a periodic orbit as the manifold boundary.

Kennedy explained his results on multiple periodic solutions for equations with nonlinearities close to step functions. The proofs begin with simplified equations whose dynamics are given by a collection of maps in finite-dimensional spaces. Kennedy also described objectives of further research.

The lectures on Tuesday and Wednesday morning dealt with numerical results, modelling and simulation. On Tuesday Humphries began with numerics. He presented results obtained with DDEBIFTOOL for equations with 1 and 2 delays, among them global Hopf bifurcation and bifurcation to invariant tori. The latter seems still out of reach of mathematical analysis.

In the next lecture Erneux explained models for chatter instabilities during deep drilling - related to the recent Deep Water Horizon disaster in the Gulf of Mexico. Other models with state-dependent delay concerned coupled lasers, malaria infection, and in greater detail, car following. The lecture also showed how to gain first insight into solution behaviour by means of heuristic use of asymptotic methods involving multiple time scales.

On Tuesday afternoon we had a further plenary session with short presentations. Walther reproduced the derivation of the equation studied by Poisson from a problem in planar geometry and suggested further work on this equation, which is not covered by present day theory since the delay involves the derivative of the solution and there are advanced arguments.

Landsman explained a model for 3 lasers, with constant delays in the coupling, and Calleja reported about his work in KAM theory.

Then we split into groups on car-following models, smoothness issues (with analyticity of periodic solutions), and population models as in Barbarossa's brief talk, and began to discuss these topics.

Wednesday morning Krauskopf spoke about delays in hybrid testing and lasers, with numerical results for some models, and discussed the nature of the delays and their relevance.

In a shorter presentation Gedeon explained gene regulatory processes in cells, with a focus on the different delays arising in transcription and transport processes.

In the afternoon we had discussions in the groups formed the day before. In the smoothness group Krisztin presented his extension of Nussbaum's result on analyticity of periodic solutions [7] to some equations with state-dependent delay.

Thursday morning and Friday morning were again devoted to mathematical analysis. On Thursday Hartung explained his recent results on differentiability of solutions with respect to parameters. This includes smoothness with respect to initial data, for the solution value in an Euclidean space and also for the solution segment in some history spaces. Such results are relevant for optimization and parameter identification problems.

Then Hupkes spoke about his work on finite-dimensional center manifolds for equations of mixed type with constant time lags and advancements. The associated initial value problem is ill-posed in general. Applications occur, for example, in modelling signal propagation along nerve fibres and in the search for travelling waves in lattice dynamical systems.

Equations of mixed type with state-dependent delays and advancements arise in the N-body problem of electrodynamics; at present this area largely is terra incognita.

The group discussions on Thursday afternoon included short presentations by Hoffman on travelling waves in unidirectional lattice dynamical systems and by Yuan on bifurcation with symmetries for a model of a ring of neurons, with constant delay in the connection terms.

In a further group we began to discuss the work of Stumpf, with the aim to relax a condition on the delay which prevents the study of the price model as a singular perturbation problem, similar to the approach of Mallet-Paret and Nussbaum [4, 5, 6].

The first lecture on Friday morning, by Magal, addressed Hopf bifurcation for non-densely defined problems. These include partial functional differential equations (with constant lags). The tool used are integrated semigroups.

Then Trofimchuk gave a shorter lecture on equations with maxima, which involve the maximum of the solution in a certain history interval. Such equations are motivated by control problems. They obviously exhibit a specific lack of smoothness. The lecture explained how the dynamics generated by an equation of this kind can be reduced to that of a system in one dimension, which is given by a non-monotone interval map. This leads to existence of chaotic solution behaviour, among other phenomena.

Following suggestions made after Trofimchuk's talk, the last afternoon began with a short lecture by Nussbaum on max-plus equations (which are not differential equations but contain transformed arguments). A theory of max-plus equations and a related eigenvalue problem was developed by Mallet-Paret and Nussbaum as a tool which helps to identify limit profiles of periodic solutions in [4, 5, 6]. Friday's lecture dealt with spectral radius results.

Next we collected problems, in a session chaired by Krauskopf. We tried to include what had been understood as a major issue during the workshop. The resulting list, formulated by Stumpf, will be posted on the website along with this report.

The remaining time was used for discussions. In the small group started on Thursday we developped a plan for proving existence of Stumpf's periodic solution without his restrictive hypothesis on the delay. The method of proof would be different, without a global invariant manifold. We hope that this meeting was the beginning of joint work on different parts of the plan of attack.

The workshop also initiated further collaborations. Krauskopf and Stumpf plan to work on numerical approximation and visualization of global invariant manifolds as in Stumpf's work, also in cases where nothing could yet be proved. Kennedy and Stumpf as well as Calleja and Krauskopf intend to collaborate. Erneux, Mallet-Paret and Nussbaum had a preliminary discussion about joint work on a two-parameter problem related to the sawtooth profile, which occurs in one of the guiding examples in [4, 5, 6]. Trofimchuk and Walther may begin to study Poisson's state-dependent delay equation from a dynamical systems point of view. Krisztin and Walther discussed the need for an introductory textbook about delay differential equations which would make the field more easily accessible to students.

The group of participants was small. In light of this there was still a great deal of heterogeneity. The workshop brought into contact various groups which had previously had little or no interaction - young researchers and senior experts, various subgroups of young researchers, and colleagues working on widely separated parts of the field, such as modelling and pure mathematics. This last aspect seems important, as differential equations with state-dependent delay require somewhat subtle mathematics which model builders should be aware of, while an understanding of the role of delays in applications may guide mathematical research in the jungle of possible relations between state histories and delays caused by these histories.

It is our hope that the workshop led some of the participants to a deeper interest in the subject and will encourage them to begin to work in this area, which is to a large extent still unexplored and full of challenging problems.

1 Diekmann, O., van Gils, S. A., Verduyn Lunel, S. M., and H. O. Walther, *Delay Equations: Functional-, Complex- and Nonlinear Analysis*. Springer, New York, 1995.

2 Hale, J. K., and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*. Springer, New York 1993.

3 Krisztin, T., and O. Arino, *The 2-dimensional attractor of a differential equation with state-dependent delay*. J. Dynamics and Differential Eqs. 13 (2001), 453-522.

4 Mallet-Paret, J., and R. D. Nussbaum, *Boundary layer phenomena for differential-delay equations with state-dependent time-lags: I.* Archive for Rational Mechanics and Analysis 120 (1992), 99-146.

5 Mallet-Paret, J., and R. D. Nussbaum, *Boundary layer phenomena for differential-delay equations with state-dependent time-lags: II.* J. für die reine und angewandte Mathematik 477 (1996), 129-197.

6 Mallet-Paret, J., and R. D. Nussbaum, *Boundary layer phenomena for differential-delay equations with state-dependent time-lags: III.* J. Differential Eqs. 189 (2003), 640-692.

7 Nussbaum, R. D., *Periodic solutions of analytic functional differential equations are analytic*. Michigan Math. J. 20 (1973), 249-255.

8 Poisson, S. D., *Sur les équations aux différences melées*, Journal de l'Ecole Polytechnique, Tome VI, cahier 13 (1806), 126-147.

16. Control and optimization of open quantum systems for information processing

June 21 to June 25, 2010

Organized by Constantin Brif, Matthew Grace, Andrew Landahl, and Wayne Witzel

Original Announcement: This workshop will be devoted to identifying and exploring hybrid methods to improve protection of quantum information processing against decoherence by integrating elements of dynamical decoupling, optimal control, and quantum error correction.

The field of quantum information involves the complex task of designing and effectively manipulating multi-qubit systems. One of the most critical difficulties besetting this task is corruption of quantum information caused by decoherence. Finding solutions to the decoherence problem, resulting from the unavoidable interaction of a quantum system with its environment, is one of the most significant challenges impeding scalable realizations of a quantum information processor. Current strategies for decoherence management were developed by researchers from three distinct communities within the quantum information sciences, namely, dynamical decoupling, optimal control, and quantum error correction. This workshop will enable researchers from these communities to identify, explore, and develop novel solutions, integrating classical and quantum resources for improved protection of quantum information processing. One crucial issue that will be at the focus of discussion is the need to maximize utilization of classical controls, which are a rich resource for the manipulation of quantum systems. By combining the expertise of each community, this workshop will address the decoherence problem from a unique interdisciplinary perspective.

The main goals of this workshop include (a) providing researchers with the collaborative opportunity to investigate the integration of methods from the fields of dynamical decoupling, optimal control, and quantum error correction and (b) identifying novel solutions with practical implementations that will decrease the error rate of quantum gates far below the threshold required for the fault-tolerant quantum computation. Since researchers in these fields have only recently begun to combine methods, this workshop will provide a crucial opportunity for an interdisciplinary forum in a largely unexplored area that could significantly impact the future of quantum information science. This necessarily involves developing hybrid methods to produce control solutions that are also characterized by a high degree of robustness to fluctuations in the control field and/or processor uncertainties.

Organizer's report:

This workshop was held at the American Institute of Mathematics (AIM) in Palo Alto, CA from June 21-25, 2010. The workshop enabled researchers from three distinct quantum-information-science research communities to interact and exchange ideas. Specifically, these fields were dynamical decoupling (Lie-algebra based), optimal control (topology/variational-calculus based) and quantum error-correction (algebraic-coding based). Participants included professors, post-doctoral fellows, and graduate students from academia as well as government and private-institution researchers. By many measures, the workshop was successful; the tutorials were interactive and enlightening and the collaborative discussion sessions were lively and engaging. Verbal feedback from the participants was universally positive and several participants reported that they formed new cross-cutting scientific relationships as a result of the workshop. In this report, we summarize the workshop's outcomes, including a list of future research directions indicated by participants.

1. Workshop Motivation

The field of quantum information involves the complex task of designing and effectively manipulating multi-qubit systems. However, this problem is beset by significant difficulties, such as corruption of quantum information caused by decoherence and under-utilization of classical controls, which are a rich resource for the manipulation and protection of quantum systems. Finding solutions to the problem of decoherence, resulting from the unavoidable interaction of a quantum system with its environment, is one of the most critical challenges impeding practical realizations of a quantum information processor (QIP). Current strategies for decoherence management were developed by researchers from three distinct communities within the quantum information sciences (QIS), namely, dynamical decoupling (DD), optimal control (OC), and quantum error correction (QEC). This workshop enabled researchers from these communities to interact and generate novel solutions. By combining the expertise of each community, the workshop explored the decoherence problem from a multi-disciplinary perspective.

DD, which has its foundations in the spin-echo effect pioneered by E. Hahn [14] and average Hamiltonian theory [30, 31] for nuclear magnetic resonance spectroscopy, seeks to decouple a system (e.g., a QIP) from its environment, thereby extending the lifetime of quantum memory channels. This is accomplished by applying a sequence of control field pulses to the system, essentially producing an approximated time reversal of the system-environment interaction. Designs of DD pulse sequences often employ only the most basic information on the system-environment interaction, without detailed knowledge of the system and environment Hamiltonians. Improvements can be added to DD by concatenating the pulse sequences [17], thereby increasing their complexity and duration. Typically, DD pulse sequences are designed *a priori*, which corresponds to the so-called "open-loop" control process; however, a "closed-loop" laboratory optimization of pulse intervals in a DD sequence was recently demonstrated [3].

OC is a general procedure employing iterative approaches such as adaptive feedback and variational calculus to design controls for any physical objective that can be formulated mathematically [7, 28], e.g., preservation of quantum memory channels or construction of high-fidelity quantum operations. It is also particularly well-suited for optimization of multiple objectives, while including constraints on admissible controls. Applying OC methods to control of quantum phenomena has developed during the last two decades into an important field of interdisciplinary research, with a rich history of theoretical developments and successful experiments [23, 2, 26, 6]. OC can be employed either in the open-loop fashion (with a model-based theoretical control design directly applied in an experiment), or in a closed-loop setup (with controls adaptively optimized in the laboratory, guided by measurement outcomes which are fed back to a learning algorithm). Recent studies strongly suggest that development and application of concepts, methods, and algorithms of OC to QIS is crucial for the progress of this field. In fact, applying OC to manage dynamics of open quantum systems and protect quantum information from decoherence has become increasingly popular in the last few years (e.g., see Refs. [35, 5, 12, 16, 32, 24, 13]). Results obtained in these works illustrate how practical quantum computing can be greatly facilitated by OC and reveal interesting physical insights through the discovery of effective control mechanisms.

Assuming that errors occur independently on individual physical qubits and/or only weak (spatial) correlations exist in multi-qubit errors, QEC employs redundant encoding of quantum information, producing logical qubits that can be restored from errors [8, 27, 18, 11]. The errors are identified by measurements on additional (ancilla) qubits, so that the information in the logical qubit is not destroyed. Once identified, the errors can be corrected. Since operations performed on the qubits are conditioned upon measurement outcomes, QEC can be considered to be a real-time closed-loop process. Like in the case of DD, codes for QEC also can be improved by concatenation, which, however, polynomially increases the resource cost in terms of the number of physical qubits and operations required to perform a fault-tolerant quantum computation (FTQC). In addition, convex optimization methods [4] were also proposed for improving the QEC performance [25, 20, 19].

As a moniker for integrative control approaches that combine the desirable properties of the methods described above, we suggest the term "hybrid quantum control" (analogous to "hybrid quantum devices" in Ref. [29]). Figure **??** expresses the primary focus of the workshop mathematically.

2. Daily tutorials

Because none of the participants are experts in all three research areas, the schedule of the tutorials for the first two days was selected to systematically educate and inform the participants and guide the initial afternoon collaborative discussions. Tutorials for the rest of the week were selected based on the needs of the workshop.

(1) Monday
   - (a) Herschel Rabitz – *"Controlling Quantum Dynamics Phenomena with Shaped Laser Pulses Acting as Photonic Reagents"*
   - (b) Rebing Wu – *"Quantum Optimal Control Landscapes: A 'Simplicity' Theory"*

(2) Tuesday
   - (a) Frank Gaitan – *"Quantum Error Correcting Codes"*
   - (b) John Preskill – *"Fault-Tolerant Quantum Computation"*

(3) Wednesday
   - (a) Daniel Lidar – *"High fidelity quantum computation via dynamical decoupling"*

   (b) Kaveh Khodjasteh – *"Dynamical Quantum Error Correction: Hamiltonian Open-Loop Quantum Control of Open Quantum Systems"*

(4) Thursday

   (a) Daniel Lidar – *"Decouple then Compute"*
   (b) Götz Uhrig – *"Universal Dynamical Decoupling"*
   (c) Todd Brun & Andrew Landahl – *"Continuous-Time Quantum Error Correction"*

(5) Friday

   (a) Robert Kosut – *"System Identification for Control and Error Correction of Quantum Information Processing Systems"*
   (b) Alireza Shabani – *"Compressed Quantum Process Tomography"*

3. Collaborative discussion sessions

(1) Monday

   (a) Optimal control and dynamical decoupling
   (b) Optimal control and quantum error correction

   Note: both sessions were lead by OC experts.

(2) Tuesday

   (a) Quantum error correction and dynamical decoupling
   (b) Quantum error correction and optimal control
   Note: continuing the discussions from Monday, both sessions were lead by QEC experts.

(3) Wednesday

   (a) Fault-tolerant adiabatic quantum computation
   (b) Experimental constraints and limitations of model assumptions
   (c) Robust formulation of quantum optimal control I

(4) Thursday

   (a) Continuous-time quantum error correction & real-time feedback control
   (b) Robust formulation of quantum optimal control II

(5) Friday

   (a) Prospects for combining dynamical decoupling, optimal control, & quantum error correction
   (b) Group summary of workshop activities & future directions

4. Open issues for future research

4.1 Integrating dynamical decoupling, optimal control, and quantum error correction

Successfully combining DD, OC, and QEC for improved control of open quantum systems represents a tremendous accomplishment for this workshop and QIS in general. We identified several possibilities for integration of these approaches. Perhaps the most straightforward possibility involves a combination of pulse-shaping by OC with the DD+QEC method recently proposed by H. K. Ng *et al.* [22]. In this context, OC could be used to design all physical controls that are required for the DD sequences and QEC operations. As a result of the workshop, OC, in general, was identified as a hardware, rather than software, component, based on the many theoretical and experimental control studies involving system-specific models. To clarify this point, consider that DD/DCG and QEC were developed from a set of general assumptions and their implementation does not require much detailed system, control, or environment information, whereas OC (numerical in practice) requires the specification of system and environment parameters to solve the relevant dynamical equations to generate control fields and fidelities for a given objective. Combining the pulse-shaping capabilities invoked by OC with the analytical formalism of DD/DCGs is another promising hybrid method. In addition to utilizing the underlying quantum control landscape structure [15] and the effective optimization algorithms for locating control fields, control theory for quantum-mechanical objectives may be reformulated to design controls that are robust to control/system uncertainties.

4.2 Continuous-time quantum error correction and real-time feedback control

After all three control strategies (DD, OC, and QEC) were introduced and discussed, participants expressed interest in continuous-time quantum error correction (CT-QEC) and real-time feedback control (RTFC). Most of Thursday was devoted to exploring these topics. After reviewing some of the contributions to CT-QEC and RTFC in the published literature [10, 1, 9, 21], we identified several open questions and some potential solutions. One of the most significant issues involves the "back-action" effect from measuring the state of a quantum system. Unlike RTFC for classical systems, modeling the effect of measuring the state of a quantum system involves a stochastic formulation of the ensuing quantum dynamics. Unfortunately, the computational time required to solve these dynamics in real time for RTFC is incompatible with the time scales of most quantum systems. Unless simplifications are made in the numerical simulations of the dynamics or limiting approximations/simplifications are incorporated, implementing CT-QEC and RTFC will be limited by these incommensurate time scales. Additional open questions CT-QEC and/or RTFC include the following.

(1) What is the optimization objective?
(2) Given the outcome/signal of a (weak) quantum measurement, what is the update rule for the control?
(3) How can CT-QEC and/or RTFC be combined with DD and other logical operations?

(4) How does one perform a threshold analysis for CT-QEC? Such an analysis is necessary to incorporate CT-QEC/RTFC into the standard FTQC framework.

4.3 Robust control strategies and multi-objective optimization

During the workshop, there also was interest in mathematical formulation of robust control strategies for QIS objectives, including methods involving multi-objective optimization. We discussed linear optimal control as the modern "workhorse" of classical control theorists and limitations on applicability of this formalism to quantum-mechanical dynamics/objectives. Specifically, we discussed the "small gain theorem" [33, 34] and limitations of linearized feedback control theory for a nonlinear system. It was during the "OC for QEC" collaborative session on Monday that CT-QEC and RTFC were initially discussed as a natural union of OC and QEC. We also considered types of objective functionals applicable for control of quantum information systems, with the goal of simultaneously maximizing both fidelity and robustness to control noise. Because maxima or minima of the objective functional determine control optimality, the development of an appropriate objective functional is important for control and optimization (for both classical and quantum systems) [28]. After two collaborative sessions on robust control theory, we formulated the optimal robust control problem as a multi-objective control, considering noise and/or uncertainty from control fields and system parameters. This formulation needs to be further explored analytically and numerically to determine the topology of the control landscape and its effect on robustness for some example problems. Future research in this direction would involve expressing the fidelity error in terms of the control field characteristics and noise spectrum. Such an expression would make it possible to optimize robustness for a given random noise process with particular spectral characteristics.

4.4 Integrating optimal control theory and fault-tolerant quantum computation

The workshop also helped to reveal the need to combine FTQC and OC. Previous work has shown that DD and QEC can enable FTQC, but the corresponding analysis for OC remains a largely open and unexplored problem. Pending the outcome of this analysis, it would be interesting to explore FTQC in the context of a hybrid "DD+OC+QEC" quantum control scheme, similar to what was done by H. K. Ng *et al.* [22]. Development of error-threshold-based objective functionals for optimal control was suggested.

4.5 Additional objectives and open issues

(1) Quantum gate fidelity balance between optimal and robust control.
(2) Benefits of random DD sequences and DD acting on the environment.
(3) Realistic quantum device engineering guided by theory.
(4) Validation for quantum devices (determining/improving error measurements and noise models), including a set of "community problems".
(5) Relevance of worst-case error versus average/probable fault-tolerance.

5. Concluding remarks

In the general field of quantum control, including the relatively new area of hybrid quantum control, much work remains! Because of the (a) demanding levels of operational fidelity required for fault-tolerant quantum computing, (b) uncertainty in system parameters, (c) noisy classical controls, and (d) unavoidable system-environment coupling, it is *crucial* to continue developing improved hybrid methods for integrating DD, OC, and QEC to successfully realizae a scalable QIP. This workshop provided a forum to establish some of the foundations of hybrid quantum control and cultivate new collaboration opportunities.

With interactive multi-disciplinary tutorial presentations in the mornings and dynamic collaborative sessions in the afternoons, this workshop provided an important service to the QIS research community. The largely extemporaneous/spontaneous structure AIM workshops is quite unique; the agenda evolved daily according the interests of the participants. Throughout the week, many new relationships for potential future collaborations were formed as a result of illuminating discussions between researchers from the different communities DD, OC, and QEC.

We express our deepest gratitude to the AIM staff, who were wonderful hosts. Their guidance and suggestions in helping us organize and run a successful workshop is greatly appreciated. We also thank all of the participants for their active involvement, especially those who presented tutorials and lead collaborative discussion sessions.

17. How to run a Math Teachers' Circle (AIM)

June 28 to July 2, 2010

Organized by Tatiana Shubin, Tom Davis, and Joshua Zucker

18. Components of Hilbert Schemes

July 19 to July 23, 2010

Organized by Robin Hartshorne, Diane Maclagan, and Gregory G. Smith

Original Announcement: This workshop will be devoted to understanding the irreducible component structure of Hilbert schemes.

Hilbert schemes, introduced by Grothendieck over fifty years ago, have become the fundamental parameter spaces in algebraic geometry. They provide a natural setting for deformation theory and play a key role in the construction of moduli spaces. Despite their importance, many basic geometric properties of Hilbert schemes remain a mystery.

Pathological examples show that Hilbert schemes can have numerous irreducible components, complicated non-reduced structures, and arbitrarily bad singularities. On the other hand, Hilbert schemes parametrizing subschemes of projective space are always connected and Hilbert schemes of points on a smooth surface are smooth and irreducible. The broad aim of this workshop is to explore the significant gap between the well-understood Hilbert schemes and the pathologies.

Working towards this goal, we will focus on the three specific problems:

- characterize the smoothable component of the Hilbert scheme Hilbd̂ (An̂) of d points in affine n -space,
- determine if the Hilbert scheme H_d,g of locally Cohen-Macaulay curves of degree d and genus g in projective 3-space is connected,
- describe the irreducible components of multigraded Hilbert schemes.

Organizer's report:

Hilbert schemes are among the most fundamental moduli spaces in algebraic geometry. Despite their importance, many of their basic geometric properties remain a mystery, and there is a significant gap between well-understood Hilbert schemes and known pathologies. The mathematical aim of this workshop was to explore this gap. In particular, the workshop focused on the Hilbert scheme $\mathrm{Hilb}^d(\mathbb{A}^n)$ of d-points in affine n-space, the Hilbert scheme $H_{d,g}$ of locally Cohen-Macaulay curves in projective 3-space, and small multigraded Hilbert schemes.

The participants included mathematicians who use a wide range of algebraic, geometric, computational, or combinatorial techniques to study Hilbert schemes. A surprisingly large number of the participants were not well-acquainted or had not met in person. A broader goal was thus to understand different perspectives, learn new methods, and prompt collaborations.

The first day of the workshop began with two survey talks. Robin Hartshorne provided an overview of the Hilbert scheme $H_{d,g}$ of locally Cohen-Macaulay curves in $\mathbb{P}^3$ and Daniel Erman reviewed the geometric properties of $\mathrm{Hilb}^d(\mathbb{A}^n)$. Both speakers highlighted open problems and key examples. In the afternoon, we formed four learning groups. One group examined the explicit equations appearing in the Ekedahl-Skjelnes' blow-up construction of the smoothable component of $\mathrm{Hilb}^d(\mathbb{A}^n)$. A second group analyzed the various possible definitions for the Hilbert-Chow morphism and, for the case of the Hilbert schemes of points, explained that these definitions coincide. A third group studied points in $\mathrm{Hilb}^8(\mathbb{A}^4)$ which lie in the intersection of its two irreducible components. The fourth group constructed some examples and counterexamples of points and paths in $H_{4,-1}$. The activities of these subgroups were reported back to the entire group by Melanie Wood, Jason Starr, Dawei Chen, and Diane Maclagan respectively.

Tuesday started with a lecture by Mireille Martin-Deschamps on tools for exploring $H_{d,g}$. She emphasized the important numerical invariants, the Rao module, extremal curves, and liaison. Milena Hering then gave a talk introducing multigraded Hilbert schemes and relating them to other Hilbert schemes. The afternoon was devoted to five working groups. One attempted to compute the tangent spaces to the smoothable component at torus-fixed points on $\mathrm{Hilb}^8(\mathbb{A}^4)$. Another group examined the moduli space of branch-varieties; introduced by Alexeev-Knutson, this variant of the Hilbert scheme replaces subschemes with finite morphisms. The next group looked for connections between Alexeev's moduli of stable toric pairs and certain multigraded Hilbert schemes. Yet another group contemplated the representation theory techniques used to understand the singularities of $\mathrm{Hilb}^9(\mathbb{A}^8)$ by Kyungyong Lee. The final group investigated families of Rao modules. Verbal summaries of each group's insights were provided by Mark Huibregtse, Roy Skjelnes, Matthew Satriano, Dustin Cartwright, and Leila Khatami.

On Wednesday, Mike Stillman spoke about local equations for a neighbourhood of a torus-fixed point on a Hilbert scheme and demonstrated how these methods can reveal the structure of $\mathrm{Hilb}^{3z+1}(\mathbb{P}^3)$ in *Macaulay2*. Following this, Ravi Vakil explained Murphy's Law for Hilbert schemes and presented several challenge problems. After lunch, we enjoyed an unexpectedly long problem session moderated by Izzet Coşkun and recorded by Li Li. The day finished with four working groups: one investigating a Grassmannian component with Hilbert function $(1, 4, 10, 6)$ of the $\mathrm{Hilb}^{21}(\mathbb{A}^3)$, one constructing particular paths on the $H_{4,-1}$, one exploring examples arising from Mnev's universality theorem, and one probing the connectedness of the moduli stack of genus g curves. Laurent Evain, Scott Nollet, Mark Huibregtse, and Li Li relayed the progress of these short-lived groups.

Dawei Chen began on Thursday with a talk on the birational geometry of Hilbert schemes concentrating on the main component of $\mathrm{Hilb}^{3z+1}(\mathbb{P}^3)$. Martin Olsson followed with a talk on Log geometry and how it can be used to distinguish certain components of moduli spaces. The afternoon was again devoted to five working groups. One determined generators for the effective, moving, and ample cones on $\mathrm{Hilb}^2(\mathbb{P}^2)$. Another group created an assortment of examples of locally Cohen-Macaulay surfaces in $\mathbb{P}^4$. The next group attempted to find the generic point on a Grassmannian component of $\mathrm{Hilb}^{21}(\mathbb{A}^3)$ be understanding the deformations of a specific ideal. A separate group worked to understand the relationship between log geometry and the toric Hilbert scheme in a concrete example. The last group investigated some determinantal equations for Hilbert schemes of points. Izzet Coşkun, Enrico Schlesinger, Anthony Iarrobino, Jesse Kass, and Erik Carlsson verbally summarized the developments for each group.

The final day started with Jason Starr speaking about pseudo Hilbert schemes and their potential applications. Mathias Lederer next described the combinatorial objects labelling the irreducible components of certain Gröbner strata of $\mathrm{Hilb}^d(\mathbb{A}^n)$. Once more, the afternoon was dedicated to five working groups. The first sought to write down the equations for four lines in $\mathbb{P}^3$ degenerating to a multiple line. The second recounted the connections between moduli of quiver representations, Hilbert schemes, and nilpotent matrices. The third endeavoured to construct toric degenerations of some Hilbert schemes. The fourth investigated specific examples of Gröbner strata in an effort to generalize results from Lederer's talk. Finally, the fifth group strived to exhibit low-dimensional irreducible components of $\mathrm{Hilb}^d(X)$ where X is a singular curve. The workshop concluded with short group reports from Robin Hartshorne, Roy Skjelnes, Daniel Erman, Margherita Roggero, and Jesse Kass.

By general consensus, the afternoon working groups were particularly worthwhile. Several participants said that this was the most enjoyable conference they had attended recently. We are grateful to AIM for providing such pleasant working conditions, and to our participants for sharing so generously of their knowledge and ideas.

19. Differentiable structures on finite sets

August 2 to August 6, 2010
Organized by Charles Fefferman and Nahum Zobin

Original Announcement: This workshop will focus on the recent activity in the study of Lipschitz structures on finite sets. Is there a reasonable notion of structures on a finite set involving higher degrees of smoothness?

A lot is known about whether a given function f on a large finite subset $E \subset \mathbf{R}^n$ extends to a C^m function on the whole of $\mathbf{R}^n$ with small norm.

For instance, suppose $f : E \to \mathbf{R}$, where E is an arbitrarily large finite subset of the plane. Assume that the restriction of f to any six points of E can be extended to the whole plane with C^2 norm less than 1. Then f can be extended to the whole plane with C^2 norm less than a universal constant.

The analogous results for Sobolev norms are at a much earlier stage. We would like to make further progress on these (and related) problems, and to explore whether there is a sensible version of these questions for finite sets not necessarily contained in $\mathbf{R}^n$.

Organizer's report:

At the Workshop there were lectures on the following topics:

1. Extension Theorems for Sobolev spaces on arbitrary sets (Fefferman, Israel, Shvartsman).
2. Algorithmic questions of smooth extensions; sparsifiers (Klartag)
3. Sobolev extension domains (Shvartsman, Zobin).
4. Connections between exotic smooth structures and physics (Lo).
5. LeGruyer's Theorem (Fefferman)
6. Linear extension operators (Luli).

In the afternoons there were discussion groups on the above topics and also on additional problems including:

7. C^∞ algebraic geometry (Nicoara)
8. Jet ideals (Fefferman, Guckenheimer, Klartag, Nicoara, Zobin).
9. Natural extension operators from domains (Goldshtein, Zobin).
10. Analogs of sparsifiers for Whitney problems (Klartag, Naor).
11. Applied problems (Guckenheimer, Yomdin).

Progress made:

There were two solutions of the Whitney extension problem for the spaces $L_p^2(R^2)$, by Israel, and by Shvartsman. The solutions are very different and the relations between the final answers are still not clear, but we have made some progress towards understanding this.

There was progress towards a solution of the problem of description of Sobolev extension domains for arbitrary smoothness in the plane; a necessary geometric condition was obtained by Zobin, and a sufficient condition was earlier obtained by Shvartsman. The conditions seem to be quite close, we hope that they coincide.

A plausible elementary inequality for polynomials arose from the discussion of jet ideals, but Guckenheimer found a counter-example.

It appeared briefly that the known results about sparsifiers were relevant to algorithmic aspects of Whitney problems, but more careful examination showed that new results on sparsifiers would be needed; the required new properties of sparsifiers may or may not be true.

The organizers would like to thank Brian Conrey and the whole AIM staff for an excellent job in making the meeting pleasant and productive.

1. Algebraic topology and physics

June 1 to June 5, 2009
Organized by Hisham Sati

Organizer's report:

The meeting: Successful attempts to span large disciplinary gaps usually involve researchers from one cutting-edge subject bringing workers in a related field up to speed on the latest advances in their subject. What seems to have been unusual in this project is that both of the groups involved have introduced powerful new methods and ideas, and that their interactions involved give-and-take at the highest level on both sides.

Many of the developing ideas in string theory come from the consideration of very subtle specific models, and part of the agenda of the meeting was to identify commonalities among these models, independent of the intricate details required by realistic physical examples. The mathematicians, on the other hand, have begun to see some very deep structures but have been unsure how to apply their insights in the service of physics.

The SQuaRE was concerned with deep notions of orientation, in both physics and mathematics. It seems to have been extremely successful in clarifying what are sometimes called higher twisted sectors in string-theoretic models (having to do with phenomena of cohomological dimension three), and it pushed these ideas to define a new research area, concerned with orientation-like structures of cohomological dimension four. These issues have been a kind of glimmer on the horizon in recent work of Witten, Moore, and others, and bringing them into focus seems to me to be a very striking new development.

New developments: There were two recent major developments immediately before the meeting. The first one is on *conformal nets*, which are higher categorical notions of Clifford algebras capturing some aspects of conformal field theory and at the same time providing a geometric model for elliptic cohomology. The group was very fortunate to have Mike Douglas from Stanford come and explain this interesting fresh development, which is directly related to the problems discussed. Kapranov and Morava immediately realized the main point and provided some constructions that build on that work.

The second development is the solution of the Kervaire invariant one problem by Hill, Hopkins, and Ravenel. Mike Hill explained the problem and the solution to the group. This was very helpful as, in addition to its deep value in topology, it seems to relate to some deep questions related to the mathematics of M-theory.

Discussions:

1. J. Morava *Quantized timelines.*

   Morava has already built on the work of Bartels-Douglas-Henriques on conformal nets to provide an interesting description of two 2-categories, one for algebras and another for one-manifolds, and explores relations to Von Neuman algebras. This is already written up and will lead to a paper.

2. M. Ando and H. Sati discussions on *M-brane charge and twisted generalized cohomology.*

   The charge of M-branes is shown to naturally live in a twisted version of TMF. This generalizes the description of D-brane charge in string theory as elements in K-theory. Although Ando was not able to attend, he and Sati met at a conference in Texas just before the SQuaRE and made a good deal of progress in solving this problem.

3. N. Ganter and M. Kapranov, discussions on *Characters of 2-categories.*

   Continuation of discussion on work in progress on 2-lambda rings and on symmetric and exterior powers of categories. While Ganter was not able to attend (for family reasons), they have already made a good deal of progress on this project that was started during the first meeting.

4. M. Hill and H. Sati (with Mike Douglas), discussion on *relating TMF to twisted K-theory via M-theory and String theory.*

   Kriz-Sati and later Ando-Sati found interesting relations between M-theory and elliptic cohomology. It is known from one hand that M-theory is related to string theory by circle bundle and from the other hand that the fields in string theory are described by twisted K-theory. Hence it is natural to ask how TMF on a total space of a circle bundle might be related to twisted K-theory on the base of the circle bundle. A concrete proposal for attacking this problem is put in place and is being discussed by email.

5. F. Redden and H. Sati, discussions on *Canonical representatives of the Fivebrane class.*

   Investigating the geometry and the harmonic representatives of the Five-brane class introduced by Sati-Schreiber-Stasheff is further explored using the analogous construction for the String class in Redden's thesis. Redden and Sati have made a great deal of progress on this problem and expect one or two papers to be written soon. A meeting between the two is planned for later this summer.

Outlook to the future: We believe this is a very fruitful program. There are ongoing various discussions which will lead to further developments. Ando, Ganter and Kitchloo were not able to attend because of family matters. There are ongoing research projects started between Ando and Sati, between Ganter and Kapranov, and between Kitchloo and Morava. Given also the interesting new developments, we are very interested in having a third meeting. That would be essential to wrap the interesting projects started in the first meeting and developed in the second meeting. We thank AIM for the opportunity and we all look forward to meet again next year.

2. Augmentations, rulings, and generating families

June 1 to June 5, 2009
Organized by Lisa Traynor

Organizer's report:

In recent years, non-classical invariants for Legendrian knots have arisen from different approaches. "Geometrical" invariants have arisen from the theory of generating families, "combinatorial" invariants have arisen from the theory of rulings, and "algebraic" invariants have arisen from the DGAs that arise through Symplectic Field Theory theory. By work of Fuchs, Sabloff, Pushkar, Rutherford, and others it is known that

$$\exists \text{ generating family} \iff \exists \text{ graded normal ruling} \iff \exists \text{ graded augmentation},$$

and that these are *not* one-to-one correspondences. The goal of our group is to understand these correspondences more precisely. On the first day of our meeting, we formed some general motivating questions. Namely,

- For a fixed Legendrian submanifold, if we use the notion of DGA homotopy classes to form equivalence relations on augmentations, what are the corresponding equivalence relations on rulings and generating families?
- How do we make the above equivalence relations extend to Legendrian isotopy classes?

During the week, Henry gave a number of explanations of his thesis work in which he has defined a notion of "decorated rulings", a sequence of Morse complexes and transition maps that can be associated to a wave front diagram. For such decorated rulings, he has defined a notion of equivalence that seems likely to correspond to the chain homotopy equivalence on augmentations. From our discussions, conjectures arose about the appropriate notions of equivalence on generating families. As goals for future work, we list the following projects based on our current conjectures:

(1) Give a simple combinatorial description of all equivalence classes of decorated rulings. There is a "standard form" of decorated rulings when all switches are marked and some returns are marked. Show that a notion of "electrical networks" can be used to determine when standard forms are equivalent.
(2) Show that there is a one-to-one correspondence between the equivalence classes of decorated rulings and the equivalence classes of augmentations. It is known that there is a surjective map from the set of equivalence classes of decorated rulings to the set of equivalence classes of augmentations; it remains to show that that this map is injective.
(3) Show that for every decorated ruling R, there exists a generating family F and a metric g so that the Cerf diagram of (F, g) (decorated with handle slides) gives R. Moreover, show that this generating family will be unique up to fiber preserving diffeomorphism and stabilization.
(4) Show that the equivalence relation on decorated rulings given by Henry and Pushkar corresponds to deforming the metric in the (F, g) pair.
(5) Show that the set of equivalence classes of decorated rulings form a combinatorial representation of an appropriate modification of the classical Steinberg group.

Once we find corresponding equivalence relations on generating families, decorated rulings, and augmentations, it will be important to see if these equivalence relations descend to the corresponding homology groups. For a fixed Legendrian knot K, for each generating family (F, g) it is possible to form "Generating Family Homology" groups, $GFH_*^{(F,g)}(K)$, and to each augmentation ϵ it is possible to form "Linearized Contact Homology" Groups, $LCH_*^{\epsilon}(K)$. Rutherford explained his ideas of forming homology groups from the combinatorial data of a decorated ruling R; these "Decorated Ruling Homology" groups $DRH_*^R(K)$ appear to be a combinatorial version of the generating family homology groups and linearized contact homology groups. The following are some questions that we aim to answer:

(1) Do chain homotopic augmentations yield isomorphic linearized contact homology groups? In other words, is $LCH_*^{[\epsilon]}(K)$ well-defined?
(2) Directly define $DRH_*^R(K)$. Do all equivalent decorated rulings yield isomorphic decorated ruling homology groups? In other words, is $DRH_*^{[R]}(K)$ well-defined?
(3) It is immediate that $GFH_*^{[(F,g)]}(K)$ is well-defined. Under the conjectured correspondences of equivalence classes, is it true that

$$GFH_*^{[(F,g)]}(K) \leftrightarrow DRH_*^{[R]}(K) \leftrightarrow LCH_*^{[\epsilon]}(K)?$$

Some /most of the above projects are very big. We formed rough subgroups of people to work on each of these projects.

We have not yet discussed the important question of how these equivalence relations extend to Legendrian isotopy classes. It would also be interesting to study higher-dimensional Legendrians. Hopefully these will be topics we can discuss at our next SQuaRE meeting.

3. The Baum-Connes conjecture and geometric representation theory

July 13 to July 17, 2009
Organized by Jonathan Block

Organizer's report:

The goal of the meeting was to explore the links that are beginning to be noticed between the Baum-Connes conjecture in operator K-theory and derived equivalence statements in algebraic geometry and representation theory. Ben-Zvi and Nadler brought expertise in algebraic-geometric representation theory, while Block and Higson have backgrounds in operator algebras and K-theory.

The Baum-Connes conjecture and the program that has evolved around it are the best-known and best-explored aspects of the operator algebra approach to noncommutative geometry. The conjecture, when specialized to Lie groups, has strong links to the tempered representation theory of reductive groups. The most obvious resonances are with the problems of parametrizing and realizing the discrete series. This is because these representations arise as isolated points in the tempered dual, and so they are recognized very simply at the level of K-theory. But recently attention has turned to the problem of connecting the Baum-Connes program to more refined and comprehensive pictures of the tempered dual. From the Baum-Connes perspective, the purpose of the meeting was to learn from geometric representation theory how best to approach this problem.

An intriguing aspect of the Baum-Connes program from the perspective of algebraic geometry is its heavy emphasis on the roles that compact subgroup can play in tempered representation theory. This is reflected for example in the prominent role of the Dirac operator on the symmetric space of a reductive group, and in connections that the Baum-Connes conjecture makes between reductive groups and their Cartan motion groups (reflecting at a K-theoretic level proposals originally set forth by Mackey).

An important first task during the meeting was the mutual education of the participants. For example, the quickest and simplest translation of the Baum-Connes conjecture into the language of geometric representation theory led to an assertion that was little more than a tautology. Some effort was needed to determine what was lost during the translation, and how it might be restored. In addition basic tools invoked by the two areas, for example continuous fields of Hilbert spaces in operator algebras versus sheaves in algebraic geometry, are quite distinct, and reconciling (or even apprehending) their contrasting features has been a challenge.

However the work of reconciling two quite distinct viewpoints was taken on with enthusiasm by the participants, and by the end of the week we had worked a type of pidgin to exchange ideas. A striking analogy began to emerge between a key approach to the Baum-Connes conjecture and the fundamental concept, due to Beilinson and Bernstein, of the localization of a representation over the flag variety. In the mid-seventies Kasparov developed a "Dirac-Dual Dirac method" to prove the (not-then-formulated) Baum-Connes conjecture and went on to analyze the method form the point of view of the flag variety (in some special cases, at least) to prove the Baum-Connes conjecture for Lorentz groups.

Kasparovs analysis seems to be an operator-K-theoretic version of Beilinson and Bernsteins mutually inverse operations of localization to the flag variety and globalization.

Our discovery of this connection, which as yet we understand only poorly, was probably the most important outcome of the meeting. But various other topics of strong potential interest were discovered.

For example Kasparov assigns to a Lie group G a representation ring $\mathsf{R}(\mathsf{G}) = \mathsf{KK}_{\mathsf{G}}(\mathbb{C}, \mathbb{C})$ consisting of homotopy classes of Fredholm representations of G. It would be very interesting to develop an algebraic-geometric analog of the Kasparov representation ring, which in terms of a convolution product on suitable coherent sheaves on the dual of G.

To give a second example of considerable interest to us, the Baum-Connes philosophy, emphasizing as it does the role of the maximal compact subgroup of a Lie group, lends support to the so-called Mackey analogy, which links the representation theory of a reductive group G to that of the semidirect product $\mathsf{G}_0 = \mathsf{K} \ltimes \mathsf{Lie}(\mathsf{G})/\mathsf{Lie}(\mathsf{K})$. The latter can beviewed as a sort of degeneration of G (it is the normal bundle of K in G). The Baum-Connes conjecture asserts that the operator K-theories associated to G and G_0 coincide. But in addition, this assertion, combined with the computations of Mackey, strongly suggest that the duals themselves coincide when G is reductive. Computations have substantially confirmed this. However the computations have, at least for now, a wholly non-geometric character. The geometric interpretation of representations via Beilinson-Bernstein localization appears likely to provide a clearer viewpoint.

Future Plans

Several concrete problems emerged from the meeting, among them:

Problem: Construct a counterpart of the localization of a G-module via continuous fields over the flag variety $\mathsf{G}_{\mathbb{C}}/\mathsf{B}$ (or the quasi-affine space $\mathsf{G}_{\mathbb{C}}/\mathsf{N}$ that fibers over it) using continuous fields rather than sheaves. Construct a realization of Kasparov's γ-element in KK-theory (arising from the Diracdual Dirac method) that corresponds to forming the space of holomorphic sections of this field. This would be a good first step towards reconciling the Baum-Connes and algebraic-geometric approaches to representation theory.

Problem: Describe minimal K-types of irreducible G-modules in terms of the support of the corresponding twisted D-module on the flag variety. Associate to such modules coherent sheaves on twisted cotangents to the support. The global sections of such sheaves are representations of the Cartan motion group: show that the construction gives natural identications of duals of G and its motion group, respecting bases given by standard modules.

Our current plans to continue discussions and further refine the formulations of these and other problems involve a number of planned meetings in Philadelphia, Chicago and Austin over the coming months of various pairs of researchers among the four of us. During the summer of 2010 we hope to arrange a week-long get-together.

4. *Ehrhart functions*

July 13 to July 17, 2009
Organized by Jesus De Loera

Organizer's report:

We report on the work done while the five team members visited the American Institute of Mathematics in Palo Alto in July 2009. The participants are grateful to AIM for the hospitality and the financial support received. We are happy to report that we made concrete progress during our visit. We intend to continue our work for at least a couple of years. An influential software package `LattE-Nouveau` will be the final product of our theoretical investigations. It is clear (judging from the users comments of previous versions) that such a package will be welcome by mathematicians and statisticians alike.

1. The Mathematical Set-Up

Let $\mathcal{P}$ be a rational convex polytope in $V = \mathbb{R}^d$ and $h(x)$ a polynomial function on V (often called a *weight function*) . We are interested to compute, or at least estimate, the sum of values of $h(x)$ over the set of integral points belonging to $\mathcal{P}$, namely

$$S(\mathcal{P}, h) = \sum_{x \in \mathcal{P} \cap \mathbb{Z}^d} h(x)$$

This problem appears often throughout all areas of mathematics. Im- portant examples of such problems appear, for instance, in enumerative combinatorics [1], statistics [11, 9], and non-linear optimization [10]. For this reason we are fairly sure our project will be of interest and have impact in a wide range of topics.

It is well-known that when the polyhedron $\mathcal{P}$ is dilated by an integer factor $n \in \mathbb{N}$, we obtain a function of n, the so-called *weighted Ehrhart quasi-polynomial* of the pair $(\mathcal{P}, h)$

$$S(n\mathcal{P}, h) = \sum_{m=0}^{d+M} E_m n^m$$

This is a quasi-polynomial in the sense that the function is a sum of monomials up to degree $d + M$, where $M = \deg h$, but whose coefficients E_m are periodic functions of $n \in \mathbb{N}$, with period the smallest integer q such that $q\mathcal{P}$ is a lattice polytope (we recommend [3, 8] for excellent introductions to this topic). The main concern of our project is the fast computation of the coefficients E_m.

Let us quickly review the current status and prior results achieved:

To begin it is important to stress that computing *all* the coefficients E_m for $m = 0 \ldots d+M$ is an NP-hard problem, thus the best one can hope for is an approximation. Until now most results dealt only with the *unweighted case*, i.e. $h(x) = 1$ and we summarize them here: Barvinok first obtained for integral polytopes $\mathcal{P}$ (vertices have integer coordinates) a polynomial-time algorithm that for a fixed integer k_0 can compute the first $k_0 + 1$ Em coefficients, that is E_m for $m = d+M$, $d+M-1$, $d+M-k_0$, (see [6]). Later, in an important breakthrough (see [7]), Barvinok proved that for fixed k_0 the k_0 highest degree coefficients of the *unweighted* Ehrhart quasi-polynomial of a *rational* simplex can be computed by a polynomial algorithm, even when the dimension d is part of the input.

Because we refer to this below let us recall that Barvinok obtained a formula relating the k-th coefficient of the unweighted Ehrhart quasi-polynomial of a *rational* polytope $\mathcal{P}$ to volumes of sections of $\mathcal{P}$. More precisely, he stated that the sum $S(\mathcal{P}, 1)$ has a natural approximation through the *intermediate* sums $S^L(\mathcal{P}, 1)$, where $L \subseteq V$ is a rational vector subspace. For a polytope $\mathcal{P} \subset V$ and a polynomial $h(x)$, define

$$S^L(\mathcal{P}, h) = \sum_x \int_{\mathcal{P}\cap(x+L)} h,$$

where the summation index x runs over the projected lattice in V/L. In other words, the polytope $\mathcal{P}$ is sliced along lattice affine subspaces parallel to L and integrals of h along the slices are added up. For $h = 1$ and $L = V$, there is only one term and $S^V(\mathcal{P}, 1)$ is just the volume of $\mathcal{P}$ itself, while for $L = \{0\}$, we recover $S(\mathcal{P}, 1)$, the number of lattice points in $\mathcal{P}$. One can also recover a similar expression for $S^L(n\mathcal{P}, 1)$ of course which gives a quasi-polynomial expression. Barvinok then introduced a particular linear combination of the intermediate sums,

$$\sum_{L\in\mathcal{L}} \rho(L) S^L(n\mathcal{P}, 1).$$

where the sum runs over all subspaces parallel to the k-dimensional faces of the polytope for $k \leq k_0$.

Finally, there are a few results regarding the weighted case. The computation of the highest coefficient is integral of h over the polytope $\mathcal{P}$. These integrals were studied at [4], [5] and more recently, with the only practical computation results, at [2].

2. Our Accomplishments while at AIM

Now we outline what was accomplished while at AIM. At the moment we are writing a paper with the details. Note that, unlike prior work, we are dealing now with the *weighted* case, i.e. $h(x)$ is a non-constant polynomial.

(1) For an integer constant k_0, we present an efficient algorithm for computing the coefficients E_m, for $m = M + d, \dots, M + d - k_0$ when $\mathcal{P} \subset \mathbb{R}^d$ is a simple or simplicial rational polytope and h is a power of an arbitrary rational linear form on $\mathbb{R}^d$. Essentially we present a short cut to the ideas of [7]. We were able to reduced the size of the set of subspaces $\mathcal{L}$ necessary, which has important practical advantage, but our approximation result is not exactly the same as in [7].

(2) Due to the representation of arbitrary polynomials as linear combinations of the powers of linear forms we derive various corollaries that guarantee the efficient computation of weighted sums for more general polynomials h, beyond just powers of linear forms (e.g. when degree is assumed fixed). Alas, from the results of [2], we know that already when h is a power of a quadratic form one cannot hope to find the desired coefficients quickly, unless P=NP.

(3) We present a generalization of the result of [7] where the poly- nomial h is replaced by and exponential function and we use the same set $\mathcal{L}$ as in [7]. We hope this more general result will find applications in the future.

(4) While at AIM we timely received the referee report for our first joint paper [2] (focusing on the first Ehrhart coefficient, which corresponds to the integration). The report was excellent and, while it requires a few more corrections, it is quite likely our paper will now appear in the prestigious AMS journal *Mathematics of Computation*. From the referee comments we feel it is quite likely that our work will also influence the area of numerical integration, at least as a way to evaluate the accuracy of existing methods.

3. Plans for the Future

We intend to continue our collaboration and if available we would appreciate very much the financial support of AIM. The exciting thing about our project is that we rely on advanced mathematical techniques to solve diffcult but important computational challenges. Already with simple `Maple` implementations we can see the excellent performance of our algorithms.

The next stage will involve a serious effort of software implementation. This will serve mathematicians in areas from combinatorics, number theory, representation theory, statistics, and many more areas. Just as the well-known ATLAS project (housed at AIM) we are sure our computational efforts will help the mathematical community. It is our intention to write a major piece of software `LattE-Nouveau` which will be the state of the art for counting and estimating the number of lattice points, as well as for computing the Ehrhart coeffcients. We will also carry on computational experimentation with our algorithms. In [2] we already reported on the practical potential of these methods for exact integration, namely the values of E_{M+d}. We are sure the impact will be great

In this next stage of the project we will involve a group of students from UC Davis (at the moment we expect 3 undergraduates and 2 graduate students will be involved in programming and experimenting). For this reason we kindly make an exceptional requests to AIM for our next visit: First, we would strongly prefer to have a longer visit (2 weeks). It appears that one week is not nearly enough for all that needs to be discussed. Second, we would like that part of that visit can take place at UC Davis. The programming team of students cannot easily be put for a week at Palo Alto and most of the computer facilities at UC Davis (with our cluster of fast servers) will allow for ease of testing. We can spend a part of the visit at Palo Alto if that is necessary. The date for the next visit is still to be decided but tentatively we could look for a date after July 2010.

[1]1 G. E. Andrews, P. Paule, and A. Riese, *Macmahon's dream*, manuscript (2009). [2]2 V. Baldoni, N. Berline, J. D. Loera, M. Köppe, and M. Vergne, *How to integrate a polynomial over a simplex*, manuscript, available at ArXiv, 2009. [3]3 A. Barvinok, *Integer points in polyhedra*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS). [4]4 A. I. Barvinok, *Computation of exponential integrals*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) Teor. Slozhn. Vychisl. 5 (1991), 149–162, 175–176, translation in J. Math. Sci. 70 (1994), no. 4, 1934–1943. [5]5 ________, *Partition functions in optimization and computational problems*, Algebra i Analiz 4 (1992), 3–53, translation in St. Petersburg Math. J. 4 (1993), no. 1, pp. 1–49. [6]6 ___, *Computing*

the Ehrhart polynomial of a convex lattice polytope, Discrete Comput. Geom. 12 (1994), no. 1, 35–48. [7]7 A. Barvinok, *Computing the Ehrhart quasi-polynomial of a rational simplex*, Math. Comp. 75 (2006), no. 255, 1449–1466. [8]8 M. Beck and S. Robins, *Computing the continuous discretely: integer-point enumeration in polyhedra*, Undergraduate Texts in Mathematics, Springer, 2007. [9]9 Y. Chen, I. Dinwoodie, A. Dobra, and M. Huber, *Lattice points, contingency tables, and sampling*, Contemporary Mathematics 374 (2005), 65–78. [10]10 J. A. De Loera, R. Hemmecke, M. Köppe, and R. Weismantel*, Integer polynomial optimization in fixed dimension*, Mathematics of Operations Research 31 (2006), no. 1, 147–153. [11]11 P. Diaconis and A. Gangolli, *Rectangular arrays with fixed margins*, Discrete probability and algorithms (Minneapolis 1993), IMA series, vol. 72, Springer, New York, 1995, pp. 15–41.

5. Arrangements, logarithmic vector fields, and D-modules

February 8 to February 12, 2010
Organized by Uli Walther

Organizer's report:

*Objects and notation

Let $\mathcal{A}$ be a central arrangement of n hyperplanes in affine ℓ-space $V = \mathbb{A}^\ell_{\mathbb{K}}$, where $\mathbb{K}$ is a field of characteristic 0. For $H \in \mathcal{A}$, denote by $\alpha_H \in V^*$ the defining linear form. Denote by $R = \operatorname{Sym} V^* \cong \mathbb{K}[x_1, \ldots, x_\ell]$ the coordinate ring of V, let $Q = \prod_{H\in\mathcal{A}} \alpha_H$ be the defining polynomial of $\mathcal{A}$, and denote by $J(\mathcal{A})$ the Jacobian ideal of Q. Let

$$D(\mathcal{A}) = \big\{v \in R^n \mid \sum_{i=1}^n v_i Q_i \in R \cdot Q\big\}$$

be the module of logarithmic derivations along $\mathcal{A}$. Here and elsewhere, Q_i is the partial derivative of Q with respect to x_i. Finally, put

$$\Omega^p(\mathcal{A}) = \{\eta \in \frac{1}{Q}\Omega^p_V \mid Qd(\eta) \in \Omega^{p+1}_V\}$$

be the module of logarithmic differential p-forms along $\mathcal{A}$. Here, Ω^p_V is the module of differential p-forms on V. The arrangement $\mathcal{A}$ is called free if one of the mutually dual modules $D(\mathcal{A})$ or $\Omega^1(\mathcal{A})$ is a free R-module.

Finished and ongoing work

Wakefield's conjecture

In our AIM meeting 2009 we succeeded in proving

Theorem 1. *If $\mathcal{A}$ is a locally free arrangement for which $J(\mathcal{A})$ has no embedded primes, and $\Omega^1(\mathcal{A})$ has projective dimension $\operatorname{pd}_R(\Omega^1(\mathcal{A})) \le 1$, then $\mathcal{A}$ is free.*

by using

Proposition 1. *If $\mathcal{A}$ is locally free, then there are isomorphisms for $p > 0$:*

$$\operatorname{Ext}^{\ell-p-1}_R(R/J, R) \to \operatorname{Ext}^\ell_R(\operatorname{Ext}^p_R(D(\mathcal{A}), R), R).$$

which is a consequence of the double-Ext spectral sequence

$$E^2_{p,q} = \operatorname{Ext}^{-p}_R(\operatorname{Ext}^q_R(M, R), R) \Rightarrow M,$$

regarding the module M as a chain complex concentrated in degree 0.

We started this year's meeting by improving this result under an additional hypothesis. Recall here that $\mathcal{A}_X = \{H \in \mathcal{A} \mid H \supset X\}$.

Corollary 1. *Assume that $\operatorname{pd}(\Omega^1(\mathcal{A}_X)) \le 1$ for all maximal non-free flats X. Then $\mathcal{A}$ is locally free if and only if $J(\mathcal{A})$ has no embedded primes away from the origin.*

In particular, an arrangement $\mathcal{A}$ is free if and only if $J(\mathcal{A})$ has no embedded primes and $\operatorname{pd}(\Omega^1(\mathcal{A})) \le 1$.

Sketch of proof. Assume that $J(\mathcal{A})$ has no non-maximal embedded primes but $\mathcal{A}$ is not locally free. Take a maximal non-free, in particular positive dimensional, flat X. Then $\mathcal{A}_X$ is locally free and $J(\mathcal{A}_X)$ has no embedded primes. Using the hypothesis, Theorem 1 then implies freeness of $\mathcal{A}_X$ - contradiction. □

Example: Let $\mathcal{A}$ be the arrangement in $\mathbb{C}^4$ defined by the polynomial

$$H = \prod(ax_1 + bx_2 + cx_3 + dx_4)$$

where the product ranges over all possible combinations of $a, b, c, d \in \{0, 1\}$ except where they are all zero. $\mathcal{A}$ is the restriction of the arrangement Edelman and Reiner found as a counter-example to Orlik's conjecture. Now let $\mathcal{B}$ be the arrangement obtained by multiplying $\mathcal{A}$ with a new variable x.

The associated primes of $J(xH) = (H, xJ(H)) = (H, x) \cap J(H)$ are all codimension 2 as the Edelman–Reiner example has no submaximal embedded components in the Jacobian. On the other hand, the free resolution of $J(xH)$ agrees with that of $J(H)$, except in degrees zero and one. Thus, the higher Ext-modules agree and so the Edelman–Reiner arrangement presents a non-maximal flat where $\mathcal{B}$ is not free.

Material in this section is being collected into a preprint.

Remarks on a result of Yoshinaga Inspired by our AIM meeting in 2009, MS improved a theorem of Yoshinaga. Namely, in dimension 3, an arrangement is free if and only if $\mathcal{A}$ contains a hyperplane such that the multi-arrangement restriction of $\mathcal{A}$ to H is free and the characteristic polynomials of $\mathcal{A}$ and the restriction relate nicely.

MS improved this result to dimension 4 without any further assumptions, and to arbitrary dimension when certain tameness conditions are in force. This is contained in the preprint [Schulze-Yoshi].

Understanding log forms

We moved next to the question under what circumstances higher log-forms are built from order one log-forms, and in case this happens how to optimally exploit it.

By definition, $\mathcal{A}$ is free if $\Omega^1(\mathcal{A})$ is so, and by duality this is equivalent to freeness of $\Omega^{\ell-1}(\mathcal{A})$. In the free case, the inclusions $\bigwedge^\bullet \Omega^1(\mathcal{A}) \subseteq \Omega^\bullet(\mathcal{A})$ become equalities of free modules. It is natural to ask whether this can happen in the non-free case. Let us write E^p for the quotient $\Omega^p / \bigwedge^p \Omega^1$. Having looked at many examples we believe that

Conjecture 1. *$\mathcal{A}$ is free if Ω^p is free for some $1 \leq p \leq \ell - 1$.*

As of yet, we can prove the following:

Theorem 2. *$\mathcal{A}$ is free if one of the following conditions holds true:*

(1) *$\Omega^2(\mathcal{A})$ or $\Omega^{\ell-2}(\mathcal{A})$ is free.*
(2) *$\operatorname{pd} \Omega^1(\mathcal{A}) \leq 1$ and $\Omega^p(\mathcal{A})$ is free for $p = 3, \ldots, \ell - 3$.*

Analyzing the proof, it turned out that one should work with certain submodules

$$\Omega_0^\bullet(\mathcal{A}) \subset \Omega^\bullet(\mathcal{A}),$$

which in a sense contain the non-trivial part of the module of log-forms. In the time shortly after the meeting, we developed the complete toolbox to work with these modules like with the modules $\Omega^\bullet(\mathcal{A})$, including the standard dualities and certain fundamental complexes. Geometrically, we could interpret the above objects as follows.

Proposition 2. *As sheaves on* $\mathbb{P}V$*,*

$$\widetilde{\Omega_0^p(\mathcal{A})} = \Omega^p(\mathbb{P}\mathcal{A})(p).$$

As a further application of our $\Omega_0^\bullet(\mathcal{A})$-theory, we are in the process of generalizing the Mustaţă–Schenck formula (the equality of the total Chern polynomial of $\Omega^p(\mathbb{P}\mathcal{A})(p)$ and the Poincaré polynomial of $\mathbb{P}\mathcal{A}$ for locally free $\mathcal{A}$) to general arrangements in $\mathbb{P}^3$.

We believe that we can further improve Theorem 2 and finally prove Conjecture 1.

Generic arrangements.

Recalling from earlier the second page

[[MATRIX]]

of the spectral sequence

$$E^2_{pq} = \operatorname{Ext}_R^{-p}(\operatorname{Ext}_R^q(M,R),R) \Rightarrow M, \tag{2}$$

we set out to see what we could learn from it in the case of a generic arrangement (*i.e.*, an arrangement of n hyperplanes in ℓ-space where any choice of ℓ hyperplanes meet exactly at the origin). We were motivated by the fact that the E_2-pages in all cases of this type that we looked at were tantalizingly simple, and we could check by hand that the E_3-pages were trivial.

Letting Ω_0^p be the kernel in Ω^0 of the contraction against the Euler derivation, one has access to some exact sequences such as the following. In

[[MATRIX]]

the first map is multiplication by the equation of the linear form f_H, and the map p is given by reducing modulo f_H.

On the other hand, recall the following result of Wiens:

Lemma 1 (Theorem 3.4, [Wiens01)**.** *] If $\mathcal{A}$ is a non-Boolean generic arrangement, then the restriction map p in* (**??**) *is surjective. Equivalently, there is a short exact sequence*

[[MATRIX]]

We proved a certain duality property about "spherical" modules (which have only one nontrivial higher Ext into the base ring):

Lemma 2. *Let R be a local ring or polynomial ring of dimension ℓ. Let M be a locally free, p-spherical module, where $0 < p < \ell$. Then $M^\vee$ is locally free and $(\ell - p - 1)$-spherical.*

With this lemma, we showed sequence (**??**) to be exact on the right for generic arrangements. As a matter of fact, we established a similar short exact sequence for higher differentials as well. Via a "straightforward" calculation, we then arrived at

Theorem 3. *If $\mathcal{A}$ is a generic arrangement of n hyperplanes of rank ℓ, and $n > \ell$, then $\Omega_0^p(\mathcal{A})$ is p-spherical, for $0 < p < \ell - 1$. Moreover, $\operatorname{Ext}_R^p(\Omega_0^p(\mathcal{A}), R)$ is an Artinian module of length $\binom{n-1}{\ell}$.*

Actually, taking more care about the cancellations one can get the actual Hilbert series:

Corollary 2. *The Hilbert series of the module $\operatorname{Ext}^p(\Omega_0^p(\mathcal{A}_{n,\ell}), R)$ is the coefficient of $s^n u^p v^\ell$ in*

$$T(s,t,u,v) = \frac{s^4 u v^3}{(1-s)(1-st)(1-st-sv)(1-s-suv)}.$$

The results described in this subsection will soon be ready to appear as a preprint.

Graphical arrangements and ghosts

Some of us (HS, GD, MW, UW) met in Japan in August of 2009. We looked at "graphical arrangements" $\mathcal{A}_G$, whose linear factors are the rows of the incidence matrix of a given graph. While computing examples of the modules $\operatorname{Ext}^i(\Omega^j, R)$ for such arrangements it became clear that there are very nice interactions between the associated primes of these modules, and the combinatorics of the graph. Basically, there seem to be two types of associated primes:

(1) primes given by chordless cycles in the graph;
(2) other associated primes (named "ghosts").

We can (easily) show that any chordless cycles of size $i+1$ is responsible for a non-vanishing Ext-module: chordless cycles of size $i+1$ induce associated primes of height i of the Jacobian ideal J and hence feature as associated primes of $\operatorname{Ext}_R^i(R/J, R)$. It is far less clear what other primes are associated to $\operatorname{Ext}_R^i(R/J, R)$.

Definition 1. *An* i-ghost *is any associated prime of the module* $\operatorname{Ext}^i(R/J, R)$ *with codimension strictly greater than i.*

Note that this definition does not depend on the arrangement being graphical and there are ghosts for non-graphical arrangements (e.g. the Edelman-Reiner arrangement).

For a given graph G, let $\overline{G}$ be the complementary graph of G (i.e. same vertices but edges are all those that are not in G). The origins of the ghosts are mysterious but we have some data that suggest the following conjecture.

Conjecture 2. *Let G be a graph and $\mathcal{A}_G$ the associated arrangement. Let $\overline{G}$ be the complementary graph of G. Let m be the length of the maximal chordless cycle in G; assume that $m \geq 4$.*

Then there is a one to one correspondence

$$\{\, i\text{-ghosts with } i \geq m \,\} \quad \Longleftrightarrow \quad \{\text{ chordless cycles of length at least } i+2 \text{ in } \overline{G}\,\}.$$

Let $D(G)$ be the logarithmic derivations on the associated arrangement $\mathcal{A}_G$. Towards understanding ghosts on a graph G we examined the following short exact sequence:

$$0 \to D(G) \to D(A_1) \oplus \cdots \oplus D(A_n) \to D(A_1) + \cdots + D(A_n) \to 0$$

where $\{A_1, \ldots, A_n\}$ is the clique complex of G. Since each of the arrangements corresponding to the A_s are free from the long exact sequence in Ext we have that

$$\mathrm{Ext}^i(\Omega^{\ell-1}(\mathcal{A}_G), R) \cong \mathrm{Ext}^{i+1}(D(A_1) + \cdots + D(A_n), R).$$

Furthermore the generators of the modules $D(A_s)$ are well understood from using invariant theory. From this we see the shift in the degree of the Ext modules. However at this point we do not see how to view the ghost associated primes from this clique complex. This is a current focus of our research.

Future projects Given that we were enjoying considerable momentum on the logarithmic forms project, we have not yet taken the time to look at several other parts of our initial proposal, although they promise to be of considerable interest. **Freeness and restrictions** Already in the ground-breaking work of Terao on free arrangements, it became clear that the naive inductive approach by removal of and restriction to individual hyperplanes is insufficient for understanding freeness. There are free arrangements with are not inductively free. However, there are more subtle relations of freeness and certain restriction maps on log-forms and log-differentials. Ziegler first showed that an arrangement $\mathcal{A}$ is free exactly if, for some fixed $H \in \mathcal{A}$, the multirestriction $\mathcal{A}^H$ is free and the restriction map $D_H(\mathcal{A}) \to D(\mathcal{A}^H)$ (from a certain submodule $D_H(\mathcal{A}) \subset D(\mathcal{A})$) is surjective. Yoshinaga showed that one can replace in this statement the restriction map by a map on log-forms $\Omega^\bullet(\mathcal{A}) \to \Omega^\bullet(\mathcal{A}^H)$, and he refined Ziegler's idea as follows: For $\ell > 3$ he showed that the above surjectivity can be replaced by the condition that $\mathcal{A}$ is locally free along H. In dimension $\ell = 3$, where $\mathcal{A}^H$ is automatically free, he showed that the above surjectivity can be replaced by an equality of characteristic polynomials $\chi_0(\mathcal{A}, t) = \chi(\mathcal{A}^H, t)$. Recently, Schulze showed that this can still be done in dimension $\ell = 4$ and even in higher dimension if $\operatorname{pd}\Omega^1(\mathcal{A}) \leq 1$. We want to investigate whether this latter condition can be dropped, and how the result can be further improved.

There is also a (different) restriction map $\Omega^\bullet(\mathcal{A}) \to \Omega^\bullet(\mathcal{A}^H)$ for $H \notin \mathcal{A}$. For generic H, this is subject to a conjecture by Ziegler, which is very closely related to Terao's conjecture. In our meeting we understood the relations between these different restriction maps better. We want to study the relation of freeness and the different restriction maps, and work on Ziegler's conjecture. **Critical sets of master functions** Some recent work of Cohen, Denham Falk and Varchenko considers critical sets of master functions $\Phi_\lambda = \prod_{i=1}^k H_i^{\lambda_i}$ for $\lambda \in (\mathbb{C})^k$. Their preprint allows one to recast freeness of $\Omega_\mathcal{A}$ in terms of the complete intersection property of the defining ideal of a certain variety $\overline{\Sigma}(\mathcal{A})$ in $\mathbb{P}^{n-1} \times \mathbb{P}^{k-1}$. Namely, the critical points of Φ_λ on the complement of the hyperplanes are defined by the n coordinate equations $d\log\Phi_\lambda = \sum_{i=1}^k \lambda_i d(H_i)/H_i = 0$. The variety $\overline{\Sigma}(\mathcal{A})$ is obtained by considering the solutions to the equations $\sum_{i=1}^k a_i d(H_i)/H_i = 0$ on the complement of the hyperplanes, where $[a_1 : \cdots : a_k]$ are coordinates in $\mathbb{P}^{k-1}$, then taking the closure in $\mathbb{P}^{n-1} \times \mathbb{P}^{k-1}$. Then $\overline{\Sigma}(\mathcal{A})$ is a complete intersection if and only if $D(\mathcal{A})$ is free; moreover, the variety is arithmetically Cohen-Macaulay if $\operatorname{pd}\Omega^p(\mathcal{A}) \leq p$ for all p. An appropriate converse statement is unknown.

It seems likely that their construction may shed some light on the properties of logarithmic forms considered above, and conversely. **Bernstein-Sato polynomials**

Bernstein-Sato polynomials and cohomology of the Milnor fiber $F_{\mathcal{A}}$ are objects of ongoing study in the singularity theory community, and their behaviour for arrangements is both extremely subtle and poorly understood. There is a known relation between the non-vanishing of Minor fiber cohomology and the zeroes of the Bernstein–Sato polynomial $b_{\mathcal{A}}(s)$ (defined as the minimal polynomial of s on the module $D[s] \bullet f^s / D[s] \bullet f^{f+1}$). The construction of $b_{\mathcal{A}}(s)$ appears closely related to the master functions described above. A recent preprint of Budur, Saito and Yuzvinsky supports this intuition and seems to indicate that one should seek a unifying relationship between syzygies of the Jacobian ideal (or, equivalently, module of logarithmic derivations), critical sets of master functions, and zeros of the Bernstein-Sato polynomial of an arrangement. Attempting to uncover this relationship is a medium-term goal for our future work.

*Summary

Our visit was highly productive; we made very good progress and we now have two papers in preparation. We are working hard on our conjectures above, as well as understanding better the connections between freeness, Bernstein-Sato polynomials, and D-modules. The workshop was great and we express our thanks to AIM for making the visit so enjoyable.

6. Hybrid optimization methods

March 1 to March 5, 2010
Organized by Katie Fowler

Organizer's report:

This focus of this collaborative effort is to develop a derivative-free hybrid optimization method suitable for mixed-integer, computationally expensive, simulation-based problems. The motivation and focus over the last two years has been on problems from hydrology including groundwater remediation problems as well as water supply problems. We refer to the algorithm, which combines the integer capabilities of a genetic algorithm with the fast, parallel local search strengths of a pattern search method, as Evolutionary Algorithms Guiding Local Search (EAGLS).

Below we describe several of our original goals, as outlined in previous reports from the original workshop and the first SQuaRE, and the efforts made during this SQuaRE. After that, we outline future work and give an overview of accomplishments from the last year. I think we can say without a doubt that the accomplishments made this week exceeded our best expectations. Even when a subset of us has been together over the last year, nothing compares to the progress made this week. We agree that many of these tasks would either go unfinished or would take months to tackle had we not all been in the same room together and been in an environment so accommodating. We are grateful to AIM and the staff for this opportunity.

To this end, we would appreciate an opportunity for a third SQuaRE during February 2011. We feel at that point that we could bring together many if the items listed in the Future Directions section listed below.

Objective

Progress

Develop a parallel, derivative-free hybrid optimization scheme for mixed-variable simulation-based problems

- Griffin has worked over the past year to improve and test the original hybrid scheme that combines the HOPPSPACK software with the NSGA-II package, developed initially at the previous SQuaRE. Gray further tested the performance of the algorithm on several benchmarking problems from hydrology. During this week, each member of the group installed and tested the software on his/her own machine. Several improvements were made to ensure the software is actually portable and user friendly. In particular, we narrowed down the input/ouput content and structure.
- The Particle Swarm Optimization (PSO) method is now included in the package but is not yet part of the hybrid scheme.

Interactive web-based optimization

- Hemker demonstrated Blackopt, which was developed during the last year at TU Darmstadt, a web-based interface to connect remote systems (black boxes) to evaluate objective functions and constraints of optimizations problems with optimization infrastructures. The system is built to handle different problems and optimization methods at one time in separate optimization jobs. It builds up in parallel a database of evaluation results to avoid function evaluation for identical inputs.

Point ranking within the EAGLS framework

- This has been a continued effort by Griffin and Parno. Sorting proposed points by their expected usefulness before performing function evaluations can help tune the hybrid structure on the fly to better handle a given problem. Currently, two point ranking methods have been implemented. The first is a ranking algorithm in place that will reward the best performing algorithm by evaluating points from this algorithm before any others, while the second ranking algorithm is based on predicting a proposed point's objective value on a radial basis function network and ranking points according to their expected function value. Initial results have shown that a dramatic decrease in the total number of function evaluations required when compared to the same hybrid algorithm without point ranking. Future work will be to implement more sophisticated ranking algorithms that take acknowledge information sharing between algorithms to estimate the usefulness of a proposed point to the entire hybrid algorithm.

Build a suite of test problems and performance measures

- Fowler and Hemker made progress on building a simulation for the Umatilla Navy Waste site. Although the transport part of the problem is not successfully running yet, we were able to implement the flow model into MODFLW2000.
- Fowler set up a water supply problem on a sequence of finer and finer simulation grids with an improved well-model to test the feasibility of previous solutions as the physical system becomes more realistic. We found that previous solutions became sub-optimal as the grid was refined and we set out to find a new solution. Gray tested EAGLS and was able to locate a new optimal point when all other methods applied failed.
- We tested a challenging mixed-variable design problem based on minimizing the volume of a compression spring. Our preliminary results show that EAGLS outperforms the published results on this problem for a PSO by using roughly an order of magnitude less function calls to locate the optimal point.
- Hemker implemented a variety of performance measures to compare algorithmic performance based on a paper by Stefan Wild (an attendee of the original AIM Workshop). In addition, since the EAGLS is a non-deterministic (i.e. stochastic) optimization method, as a group we discussed and developed performance measure to account for multiple optimization runs and constraint violations .

Future Directions

- Analyze convergence properties of EAGLS based on the non-smooth Calculus approach in a paper by C.T. Kelley and possibly by J.E. Dennis (both attendees of the original AIM workshop) for derivative-free methods
- Incorporate the Particle Swarm method as part of the hybrid scheme and investigate other possibilities (Nelder Mead, Divided RECtangles) that are readily available
- Implement the point-ranking scheme developed by Parno and Griffin into the EAGLS framework. This will be the focus of Parnos talk at the Copper Mountain Conference on Iterative Methods later this month.
- Test and analyze the possible stopping criteria and algorithm parameters to offer guidance to the user for various types of problems (for example, high dimensional, real variables only, linear vs. nonlinear constraints, low computational expense)
- Investigate the advantages and disadvantages of incorporating surrogate functions into the EAGLS framework. Hemker and Parno already made progress on this during the week in the context of a PSO by using a signal to noise ratio. This work will be submitted as part of a current research effort to Engineering Optimization.
- Parno and Gray have discussed incorporation of statistical methods for uncertainty quantification.
- Fowlers PhD student will be visiting Gray in Summer 2010 to implement the hybrid ideas into DAKOTA, an open source software (http://www.cs.sandia.gov/DAKOTA/software.html)

The following activities can be attributed, at least in part, to the collaborations formed at the AIM workshop and subsequent SQuaRE:

Papers/Publications:

Gray, G. A., Griffin, J. D., and Fowler, K., "A Hybrid Optimization Scheme for Solving the Hydraulic Capture Problem with an Unknown Number of Wells," Proceedings of the Conference on Soft Computing Technology in Civil, Structural, and Environmental Engineering, Funchal, Madeira, Portugal, September 2009.

Gray, G. A., Griffin, J. D., and Fowler, K., "Hybrid optimization Schemes for Simulation-Based Problems," accepted for the Proceedings of the International Conference on Computer Science 2010. (Reviewed with an acceptance rate of 30%) .

Thomas Hemker, Derivative Free Surrogate Optimization for Mixed-Integer Nonlinear Black Box Problems in Engineering, Reihe 10: Informatik/Kommunikation, Nr. 797, (VDI Verlag, Dsseldorf, Germany)

B. Zhang, K.R. Fowler, M. Grace, S. Mondal, S.J. Grimberg, Optimization of Anaerobic Digestion Model No. 1 (ADM1): Simulation of Dairy Manure Digestion, Proceedings of the American Society of Agricultural and Biological Engineers Annual International Meeting, Reno, Nevada, June 2009.

T. Hemker oversaw the Bachelors Thesis (TU Darmstadt, Dept. of Computer Science) work of Florian Nöll, completed in 2009, entitled "Blackopt efficient coupling of distributed black box evaluations and optimization infrastructures." The ideas were generated during the first meeting of the SQuaRE.

T. Hemker oversaw the Diploma Thesis (TU Darmstadt, Dept. of Mathematics) of Christian Werner, completed in 2009, entitled "Lipschitzian optimization supported by local search on surrogates." The idea was discussed at the AIM workshop in 2008.

Talks:

Gray, G. A., "Analysis of Uncertainty Using Optimization Iterates," SIAM Conference on Applied Linear Algebra, Monterey Bay-Seaside, CA, October 2009.

Gray, G. A., Griffin, J. D, and Fowler, K., "Solving Practical Problems with Hybrid Optimization Approaches," INFORMS Annual Meeting, San Diego, CA, October, 2009.

Gray, G. A., Griffin, J. D., and Fowler, K., "A Hybrid Optimization Scheme for Solving the Hydraulic Capture Problem with an Unknown Number of Wells," Conference on Soft Computing Technology in Civil, Structural, and Environmental Engineering, Funchal, Madeira, Portugal, September 2009.

Gray, G. A., Griffin, J. D. and Fowler, K., "Derivative-Free Hybrid Optimization Approaches to Hydraulic Capture," International Symposium on Mathematical Programming, Chicago, IL, August 2009.

Gray, G. A., Griffin, J. D., and Fowler, K., "Tailoring Hybrid Optimization Methods for Water Resources Management," SIAM Conference on Mathematical and Computational Issues in the Geosciences, Leipzig, Germany, June 2009.

Fowler, K., Parno, M., and Hemker, T., "Particle Swarm Optimization with Surrogate Functions for Water Management Problems," SIAM Conference on Geosciences, Leipzig, Germany, June 2009

Gray, G. A. and Griffin, J. D., "A Hybrid Optimization Scheme for Combining Pattern Search and Gaussian Process," seminar, TU-Darmstadt, May 2009.

Parno, M., Fowler, K., and Hemker, T., "The Use of Surrogate Functions for Hybrid MINLP Particle Swarm Optimization," SIAM Conference on Computational Science and Engineering, Miami, FL, February 2009.

Other:

Fowler and Gray co-organized a two part minisymposium entitled "Simulation-based Optimization in Hydrology," for the SIAM Conference on Mathematical and Computational

Issues in the Geosciences, June 2009 in Leipzig, Germany. Many of the speakers had participated in the AIM workshop.

Fowler received funding for a project entitled "Parameter Estimation Study for 1-D Heat Transport in the Shallow Subsurface," to collaborate with Dr. Owen Eslinger at the U.S. Army Corps of Engineers.

Gray, Fowler, and Shawn Matott prepared a proposal to SERDP for FY10 (unsuccessful but good feedback received on how to make improvements).

Gray prepared a proposal to the DOE for an ARRA Early Career award. It included extensions to the hybrid ideas. (unsuccessful, but positive technical feedback and good comments regarding AIM involvement)

Hemker visited Fowler at Clarkson University in November 2009 for one week, during which time Parno also visited campus from MIT. The objectives of this collaboration were:

(1) implementing the Umatillo Navy Waste site into MODFLOW and MT3DMS (the groundwater flow and transport simulators),
(2) outline future directions for constraint handling with derivative-free optimization methods,
(3) make revisions to a journal paper to be submitted to be submitted to Engineering Optimization regarding using surrogate functions for particle swarm optimization (with an application to a groundwater remediation problem),
(4) organizing a suite of benchmarking test problems.

Upcoming:

Gray, The corresponding talk to the proceedings paper at ICCS (May-June 2010)

Gray, Talk at Computational Methods in Water Resources (June 2010)

Parno and Gray will each give talks at the Copper Mountain Meeting on Iterative Methods (April 2010)

Gray, Fowler, and Eslinger will co-edit a special issue of the Pacific Journal of Optimization with results from the AIM workshop and SQuaRE.

Papers are in preparation for SISC, Pacific Journal of Optimization, Engineering Optimization, and International Journal of Mathematical Modelling and Numerical Optimization.

Fowler will be on sabbatical spring semester 2011 and plans to visit Gray and Griffin.

Fowler's PhD student, Jacob Orsini, received funding through the Computational Science Research Institute to visit Sandia National Labs in summer 2010. He will be implementing the hybrid optimization scheme currently being developed into the DAKOTA framework so the tool is available for public use.

7. Isometric embedding of higher dimensional Riemannian manifolds

March 8 to March 12, 2010
Organized by Marshall Slemrod

Organizer's report:

First we recall the conceptual issues in the case of a 2 dimensional Riemannian manifold. In this case a little experience with multidimensional calculus and everyday surfaces make the issues easy to understand. In lower level courses we introduce the concept of a 2 dimensional surface usually sitting above the u,v plane and write the equation for the surface as z=z (u,v). We then show students that Pythogoras's Theorem allows us to find distances along the surface is terms a distance differential "ds" where ds is expressed in terms of partial derivatives of f. Hence we have shown that given a 2 dimensional surface lying in 3 dimensional Euclidean we can find a metric measuring distance along the surface. But what about the inverse problem? Given a metric g ij dxi dxj is there always a surface? This is the isometric embedding problem. Of course one way to attempt to solve this problem is just write down the equations relating z(u,v) to gij. But a quick look at these equations shows they are fully nonlinear, squares of the partial derivatives of f appear, and hence the equations are even worse than the quasilinear equations usually occurring in applied mathematics, physics, and engineering. Now we are faced with a choice: work with the fully nonlinear system or find an equivalent problem which is quasilinear. Since applied mathematicians have quite a bit of experience in mathematics of quasilinear systems of partial differential equations we have chosen the second avenue of attack. In the case of the 2 dimensional embedding problem the relevant theorem is the Fundamental Theorem of Surface Theory which tells us that if a system of consistency or integrability relations expressing the equality of cross partial derivatives of z then we indeed have existence of a surface z=z(u,v) and z satisfies the usual relations consistent with the given metric gij.

This quasilinear system is called the Gauss-Codazzi system. It has the rather standard properties one would associate with equations of steady compressible gas dynamics, i.e. the equations have regions of hyperbolicity, parabolicity, and ellipticity depending on the Gauss curvature. (In fact this not a coincidence since Chen, Slemrod, and Wang have shown that in fact with suitable identification the Gauss-Codazzi equations in for a 2 dimensional surface are precisely the equations of steady compressible gas dynamics). Hence depending on the Gauss curvature one may attempt to use the mathematics of hyperbolic,parabolic, or elliptic partial differential equations to prove existence, hopefully global existence if possible, to the original embedding problem.

While the 2 dimensional case described above is rather well understood the case of higher dimensions is much less developed. Here the first problem to consider is the case of embedding a 3-dimensional Riemannian manifold into 6 dimension Euclidean space. While its hard for most people to picture a 3 dimensional Euclidean manifold in 6 dimensional space one way to think about the problem is to think of a 3 dimensional body of material with original coordinates u,v,w. Then the body is deformed so that a point in the body is now described by x=x(u,v,w), y=y(u,v,w), z=z(u,v,w), u,v,w which clearly generalizes the concept of a point of the 2 dimensional surface z=z(u,v), u,v. Hence it's obvious from the example that 6 is the dimension of the relevant target Euclidean space.

How about the embedding PDEs? Again a choice must be made whether to go fully nonlinear or quasilinear. We once again elect the quasilinear route and attempt to use what are now the Gauss-Codazzi-Ricci equations to prove embedding. Now there are existing results that have proven local embedding , the first in fact due to Robert Bryant, Phillip Griffiths, and Deane Yang. These papers use rather heavy machinery and part of the goal here is to exploit the known properties of quasilinear hyperbolic PDE to obtain a simple straightforward local existence result which could then be easily generalized to even higher dimensions. One more point going back to analogy with gas dynamics in the 2 dimensional manifold case: are there any points of ellipticity where perhaps elliptic theory could be used? In fact there is a theorem of Chern that states: there are no points of ellipticity for higher dimensional manifolds! This does not mean every point is hyperbolic since systems of PDEs can have some subset elliptic and some subset hyperbolic, e.g. the Euler-Poisson equations. In fact our method of proof is an attempt to show there are points of hyperbolicity.

At first we thought that the search for hyperbolity could directly follow the known method of gas dynamics, elasticity, etc. and search for what is called an "entropy" equation. That is an additional equation implied by Gauss-Codazzi-Ricci where the "entropy" is convex in the state variables. If this sounds like thermodynamics it's because in classical gas dynamics the "entropy" equation is indeed the equation governed by the physical entropy. (We note that if our approach succeeds we will have added a new concept to differential geometry, the idea of "entropy", which seems to us another nice connection between what appear to be disjoint subjects: geometry and thermodynamics). It was comparatively easy to find an "entropy" but it was not convex. So we left AIM a little disheartened but in the last days we have found what appears to be a convex quantity but does it satisfy a conservation law? It certainly satisfies an evolution equation but the equation needs to be put in so called conservation form. Three of us will be meeting in late April is South Korea to see if this can be done. Stay tuned!

Finally we note the AIM facilities and staff were as usual fantastic.

8. Algebraic K-groups

March 22 to March 26, 2010
Organized by Michael Hill

Organizer's report:

1. Participants

At this meeting, we had 4 of the 5 people named in the the original proposal. Vigleik Angeltveit (PhD MIT 2006), from University of Chicago; Andrew Blumberg (PhD UChicago 2005), from University of Texas; Teena Gerhardt (PhD MIT 2007), from Indiana University; and Michael Hill (PhD MIT 2006), from University of Virginia attended. Tyler Lawson (PhD Stanford 2004), from University of Minnesota, was unable to attend. While there had been several previous collaborative efforts (Angeltveit-Hill-Lawson have several papers on THH and Angeltveit-Gerhardt have several papers on TC), there have been no broader collaborations that link all of the participants. We are happy to say that already as a result of this meeting, we have several papers we are currently working on.

2. Progress

At the start of the SQuaRE, we created a list of 5 problems that we wanted to focus on. These were:

(1) Provide a conceptual description of cyclotomic (equivariant) structure on the topological Hochschild homology (THH) of Thom spectra, and use this to study the topological cyclic homology (TC) of Thom spectra. This would vastly generalizing the previously known cases of suspension spectra of loop spaces, provide the first description of the TC of MU, and provide a new approach to the computation of TC of $H\mathbb{Z}$.
(2) Describe the possible "false starts" for construction the spectrum TC and explain why the various seemingly natural constructions fail to produce the correct objects. For background, we note that this is an area of the foundations of trace methods which is notoriously poorly understood and broadly regarded as unsatisfactory due to its model dependence. This project will produce a document of pedagogical value for newcomers to the area as well as highlight the deep foundational issues in equivariant stable homotopy theory that are exposed by the problems here.
(3) Understand the "restriction map" in the construction of TC and describe a category of cyclotomic spectra.
(4) Produce a relative version of TC, working not over the sphere spectrum S^0 but rather over a more general E_∞-ring spectrum R.
(5) Explore the interplay between the equivariant filtration spectral sequence for TF and the restriction map, thereby finishing the computations of the algebraic K-groups of $\mathbb{Z}/p^n$ and $\mathbb{Z}[C_2]$.

These problems are highly interrelated, and much of our time at the SQuaRE was spent teasing apart the connections and dependencies. We additionally formulated a conjectural solution to the first part of Problem (1) early in the week, and by the end of the week, we had a solution which we are currently writing up. As an added bonus, our solution has a natural “relative” version, giving at least a partial answer to Problem (4). Part of our future projects include building other relative versions that might apply more generally.

We additionally spent some time looking at the specific computational aspects of the theory, working in some detail on Problem (5). Here we posed a conjecture for the nature of the restriction map, and narrowed the scope of the problem to understanding several space level constructions. An analysis of these space level constructions (and the subtleties therein) is related closely to Problem (2), since it involves the passage from an equivariant spectrum to its fixed points.

3. Planning

Since this group was so successful, we have already planned another meeting in July. We intend also to meet once more during the calendar year (possibly in December), and then to have another meeting at AIM.

On a practical note, we have set up a GoogleWave to facilitate ongoing communication and progress on our projects. This had seen a frenzy of discussion back and forth, and we intend to have a group conference call in mid-April to discuss the status of the papers.

9. Higher divergence functions

April 26 to April 30, 2010
Organized by Moon Duchin

Organizer's report:

The group of five participants met at AIM from April 26-30, 2010 to continue work on higher divergence functions in groups. For many of us, the key goal for this meeting was to further explore divergence in the mapping class group of a surface.

We spent a fair amount of time discussing various cell complexes that provide geometric models for the mapping class group, especially the construction by Harer that is based on the complex of arcs on a surface.

We made progress on the following specific goals:

- Show that the k-dimensional divergence of the mapping class group of a genus g surface is at most r^{2k+2} for $g \gg k$.
Method: Embed Harers spine in Teichmuller space. Find moderate-length curves; successively enlarge the length threshold to find more and more. Twisting around those "pushes" cells far from a basepoint, so can construct a combinatorial correspondence which amounts to an actual pushing map (continuous map from complex to outside of ball).
- For sufficiently high genus, show that the 1-dimensional divergence is at least r^3. (We suspect that in general the bound in the previous bullet point is not sharp.)
Method: use a large number of partial pseudo-Anosovs with disjoint support, then use bounds on projection to their axes in curve complexes of their subsurfaces.
- Show that Teichmuller space has exponential filling in dimension $\xi - 1$, where $\xi = 3g - 3 + n$ is the complexity of the surface.
Method: There is a quasi-flat of maximal dimension given by the thin part where a full pants decomposition is short. This looks basically like a product of hyperbolic planes by the Minsky Product Regions Theorem. Now you can make Sol-like spheres that are hard to fill, following the Epstein-Thurston outline.
- Show that the mapping class group also has exponential filling in this critical dimension.
Vague method: play with Mess subgroups. Try to break the problem down into pieces with known rates of filling.

We have been continuing our work on these questions since April over email, and we hope to write a second paper in our collaborative work.

Some other goals discussed in April are off to the side a bit:

- Understand Macuras construction of modified RAAGs with arbitrary polynomial degree in their divergence of geodesics. Use these to find arbitrary polynomial degree in higher dimensions.
- Study the two-dimensional Dehn function of Sol_7.

10. Directional discrepancy

May 3 to May 7, 2010
Organized by Dmitriy Bilyk

Organizer's report:

The SQuaRE week "Directional discrepancy," held May 3-7, 2010, turned out to be to be extremely productive and successful. In the course of the week we have obtained new results, generalized our prior work, made considerable progress in several questions, and opened up new directions for further research.

The ultimate goal of this SQuaRE program is to establish precise quantitative connections between the discrepancy estimates and the geometry of the underlying sets. In particular, we are striving to understand how the discrepancy of finite point sets with respect to rotated rectangles depends on the set of rotations, its structure and size. We are thus planning to fill the gap between the two classical situations: no rotations (in which case the estimates are logarithmic in N, the number of points) and all possible rotations (polynomial estimates, of order $N^{1/4}$).

In their prior joint work [4], the participants of the program have obtained upper discrepancy bounds for several specific types of the rotation set Ω: lacunary sequences, lacunary sets of finite order, sets with given upper Minkowski dimension. Examples of point sets providing these upper bounds are constructed as rotations of the integer lattice. The existence of a rotation, which is 'good' with respect to all the directions of Ω, are based on delicate diophantine lemmas, which present independent interest in number theory [5], [7]. We have obtained our results by extending the diophantine estimates to our setting and building on the earlier work of Beck-Chen [3] and Chen-Travaglini [6].

In the first couple days of the program, we have been able to generalize these arguments to arbitrary sets of directions and give an answer in terms of the covering function $N(x)$ of Ω – the minimal number of intervals of length x needed to cover Ω. We have in particularly proved, that, if one sets $F(x) = x \cdot N(x)$, then there exists θ such that for all $\varphi \in \Omega$ we have

$$\left|\tan(\theta - \phi) - \frac{p}{q}\right| \gtrsim F^{-1}\left(F^{-1}\left(\frac{1}{q^2}\right)\right)$$

for all p, $q \in \mathbb{Z}$. This estimate agrees with the bounds for all the particular examples that we have considered previously. Using the Erdös-Turan inequality and the machinery developed in [3], [6], [4], this lemma can be translated into the discrepancy estimates for any given set of directions.

In addition, we have shown that our prior estimates are sharp in the class of rotated lattices, at least in the case of the Minkowski dimension. More precisely, we proved that if τ_0 is the infimum of numbers $\tau > 0$ such that the angle θ satisfies $|\tan\theta - p/q| \gtrsim q^{-2-\tau}$ for all p, $q \in \mathbb{Z}$, then the discrepancy of the integer lattice rotated by θ, with respect to the axis-parallel rectangles, is at least of the order $N^{\frac{1}{2} - \frac{1}{2(\tau+1)} - \varepsilon}$. Thus if the diophantine inequality holds for $\tan(\theta - \phi)$ for all $\phi \in \Omega$, one can deduce the lower bound for the directional discrepancy. This gives us sharpness of the discrepancy estimates in terms of Minkowski dimension and leads to several new questions that we have discussed extensively: Can we prove similar lower bounds for the case of logarithmic discrepancy? Can we generalize them to arbitrary sets, in a way similar to upper bounds? Can we show sharpness of the diophantine inequalities that we are using? (All these questions require a much more delicate analysis) When does the rotated lattice stop to be the optimal set from the point of view of discrepancy (it is easy to see that for all rotations its discrepancy is $N^{1/2}$, while the optimal order is close to $N^{1/4}$, [1], [2]). The participants have developed deep intuition and understanding of these questions and are currently working on them.

There are several other directions in which partial progress has been made. First of all, we have made an attempt to generalize our results to higher dimensions. While this issue is far from being resolved, we seem to have found the right approach and language – considering general lattices, see the work of Skriganov [8]. In the case of axis-parallel rectangles, this setting translates easily into the setting of rotated integer lattices in dimension $d = 2$. However, in higher dimensions one can prove discrepancy bounds for general lattices in terms of their quantitative charateristics, even though the absence of higher-dimensional analogues of continued fractions prevents one from doing the same for rotations of the integer lattice. We believe that a careful investigation of the constants arising in Skriganov's proof can shed light on the directional discrepancy. In particular, we can already reprove the results of Chen and Travaglini concerning finitely many rotations. Incidentally, the connection to the Littlewood conjecture in number theory has come up in the discussion of this approach.

Problems discussed by the group during the program include estimates for polygons with infinitely many sides and for general sets (we have obtained some bounds, which we believe can still be improved), general lower bounds, connections to other problems (the circle problem, integer points on surfaces etc.) among many others. While at present, the progress in these directions is too marginal to report, the participants now have a substantial interest and expertise in these questions generated by the discussions during the SQuaRE week and keep working in this direction.

Unfortunately, William Chen was unable to join us for the program this year (immigration/security issues). His expertise would have been of tremendous help and importance to our investigations and discussions. We sincerely hope that the issues will be resolved and he will be able to join us next year.

All of the participants would like to thank the American Institute of Mathematics for its hospitality and a unique research environment that it has provided. Being able to spend a week together, working on mathematical problems 'from dusk till dawn' has proved to be a great stimulus for collaborative research. The discussions continued during coffee breaks, lunches, dinners and strolls around Palo Alto. The quiet and cozy atmosphere of the Institute, with its couches, library, ample snacks, coffee and drinks made the participants feel almost at home. All these factor have definitely contributed to the success of our SQuaREs week. We are currently writing one paper based on the results of the program, and, hopefully, more will be on the way. All the participants are enthusiastic and eager to continue this project and are looking forward to coming to AIM again next year.

[1]1 József Beck, *Irregularities of distribution I*, Acta Math. 159 (1987), 1–49. [2]2 ——, *On the discrepancy of convex plane sets*, Monatsh. Math. 105 (1988), 91–106. [3]3 József Beck and William Chen, *Irregularities of point distribution relative to convex polygons III*, J. London Math. Soc. 56 (1997), 222–230. [4]4 Dmitriy Bilyk, Xiaomin Ma, Jill Pipher, and Craig Spencer, *Directional discrepancy in two dimensions*, preprint. [5]5 J. W. S. Cassels, *On a result of Marshall Hall*, Mathematika 3 (1956), no. 3, 109–110. [6]6 William Chen and

Giancarlo Travaglini, *Discrepancy with respect to convex polygons*, J. Complexity 23 (2007), no. 4-6, 673–722. [7]7 H. Davenport, *A note on diophantine approximation II*, Mathematika 11 (1964), 50–58. [8]8 M. M. Skriganov, *Constructions of uniform distributions in terms of geometry of numbers*, Algebra and Analysis 6 (1994), no. 3, 200–230.

11. Geometry and physics of the new topological recursion

May 3 to May 7, 2010
Organized by Motohico Mulase

Organizer's report:

The main goal of the SQuaRE Meeting was to establish a geometric foundation of the newly discovered topological recursion formula in random matrix theory and topological string theory. More specifically, the group investigated a generalization of the geometric techniques of Eynard, Mulase, Safnuk and Zhou that may be suitable for solving the general BKMP conjecture on the Gromov-Witten theory of toric Calabi-Yau 3-folds.

The conjecture due to physicists Bouchard, Klemm, Marino, and Pasquetti states that the Gromov-Witten theory of a toric Calabi-Yau 3-fold X and the B-model theory based on the topological recursion of Eynard and Orantin, with the mirror curve C of X as its spectral curve, are equivalent. In particular, the topological recursion on C essentially calculates both the open and closed Gromov-Witten invariants of X. This conjecture has attracted the attention of various mathematicians and physicists in these days.

Bouchard and Marino have shown that when one lets the integer parameter (the framing parameter) of the simplest case of the BKMP conjecture (the framed vertex case) to infinity, then the conjectural formula reduces to a concrete, previously unknown, recursion formula for Hurwitz numbers in algebraic geometry. This is the Bouchard-Marino conjecture.

The Bouchard-Marino conjecture was solved by Eynard, Mulase and Safnuk in Fall 2009. Then the method developed by them was used to solve the BKMP conjecture for the framed vertex case by Jian Zhou, and independently also by Lin Chen, a few months after the circulation of the EMS paper.

The expectation at that moment was that solving the general case may not be so far away. To date, however, no proof, or any geometric understanding of the mechanism, has been established, despite of the efforts by many researchers.

At this SQuaRE meeting, the participants examined the current stage of developments. Eynard reported his refined theory of topological recursion. Zhou presented his solution to the simplest care of the BKMP conjecture. We then identified what would be necessary to carry the known methods to the general case.

For the cases of Hurwitz numbers and framed vertex, the conjecture was mathematically rigorously formulated in the language of algebraic geometry. But for the general case, concrete mathematical formulation is still missing. After the week we have come up with the understanding of what should be the "spectral curve" of the B-model theory for a given toric Calabi-Yau 3-fold, and which breakthrough had to be made so that the rest of the conjecture could become an automatic consequence.

The recently solved cases have a common mathematical structure. The geometric side has a combinatorial structure that satisfies the cut-and-join equation, which is a nonlinear partial differential equation for the generating function of certain Gromov-Witten invariants. The Laplace transform of the cut-and-join equation then becomes a holomorphic recursion formula for the mirror partner of the generating function, thus proves the BKMP conjecture. Here the Laplace transform serves as the mirror map.

Still the same combinatorial structure exists for the general case, and the technique of Laplace transform should apply. What is missing is the identification of the spectral curve on which the Laplace transform should be performed. This problem is identified also from a different angle as the identification of the initial condition. The topological recursion is a recursive formula, and as such, it does not determine the initial condition by itself.

At the end of the week we have come to the idea how to capture the initial condition (i.e., the spectral curve).

The week was too short for us to completely solve the proposed conjecture. We believe by the time we meet again, we should be able to carry out all the necessary mathematical hurdles to solve the BKMP

For every member of the SQuaRE, this was the first experience. We all extremely enjoyed the stimulating discussions. Although we did not solve the conjecture, numerous new ideas to generalize the theory into different directions have emerged through our discussions. These ideas will be carried out in a near future.

The four senior participants have completely different research background: string theory, statistical mechanics, integrable systems and differential geometry. Therefore, for the first few days we needed to spend some time to understand each other, because there was no common language among us.

From this point of view, the SQuaRE really functioned well and served an unforeseen purpose: to make researchers of totally different background to be able to scientifically communicate and to formulate possible theorems. Without the AIM meeting the language barrier would have prevented the two sides (physics and mathematics) to communicate.

Zhou concluded: “I had a great experience at the AIM, and was extremely impressed with the format of the SQuaRE. Indeed, I will introduce this same format at the newly established Tsinghua Center of Mathematics in Beijing.”

12. The Kadison-Singer problem

May 10 to May 14, 2010
Organized by Pete Casazza

Organizer's report:

1. Introduction

On May 10-14, 2010, B. Bodmann (University of Houston), Peter G. Casazza (University of Missouri), Vern Paulsen (University of Houston), and Darrin Speegle (St. Louis University) held their third SQuaRE on the Kadison-Singer Problem at the AIM headquarters in Palo Alto.

We arrived at AIM with a new tool for use on this problem which was developed before we arrived: *syndetic sets*. These are a generalization of *arithmetic progressions*. We felt this was our best direction for a counter-example to KS - actually *The Feichtinger Conjecture* which is equivalent to KS. It is known that FC cannot be accomplished in exponential frames by arithmetic progressions. We showed that if FC can be accomplished at all, then it must be achieved with syndetic sets. So the main question is whether we can show that the techniques which yield that FC cannot occur in general for arithmetic progressions, can be generalized to show that it cannot occur for syndetic sets. We spent three days developing the theory of syndetic sets and trying to generalize the techniques used for FC for arithmetic progressions to syndetic sets. Although we made some non-trivial progress on this, we were confronted by a fundamental problem in exponential frames - which would take a serious amount of time to address.

So we switched to using our new tool on a (apparent) weakening of FC. FC asks for unit norm Bessel sequences to be partitioned into a finite number of *Riesz basic sequences*. There is a concept which is slightly weaker than a Riesz basic sequence: ℓ_2-independent sets. A weaker form of FC asks if Bessel sequences can be partitioned into a finite number of ℓ_2-independent sets? No progress has ever been made on understanding how this apparent weaker form of FC relates to FC - despite a serious amount of work having been put into this idea. We made a breakthrough on this problem which surprised all of us. It was so surprising that Speegle was in charge of typing it up and put as a title on the notes: *The awesomest theorem ever*. What we showed was that FC actually holds for this weaker form for exponential frames and in a really spectacular way. i.e. Not only can we partition our sets into ℓ_2-independent sets, but we can do it so that they are all spanning. We will spend this year developing this theorem and its consequences - which we expect to be quite significant. This really is an exciting development since until now we thought that the weaker form of FC might actually be equivalent to FC.

2. The Third SQuaRE

Since our SQuaRE was the first to have three meetings, AIM asked me to comment on the value of the third meeting. i.e. Is this really worthwhile? This is actually not so easy to say. The problem is that I only had one criterion for our meetings: to solve the Kadison-Singer Problem. Since we did not, I am not willing to say we were successful. However, if I remove this from the equation, I would have to say that at each of our three meetings, we produced a significant advance for the problem and for frame theory in general. These are some of the deepest and most important papers in the field. They could not have been written without the deep concentration on the problem provided by AIM. So I can conclude that AIM has been a major force in advancing the field and on advancing the Kadison-Singer Problem. The results we obtained in the third meeting really are quite a surprise and probably will be the strongest results which will be true after we eventually give a counter-example to the general problem. We are very excited about these developments and their possible implications. Also, the work we did on syndetic sets still represent the best direction for a counter-example for KS and so this is also an important development from our meeting.

So my conclusion is that the outcome of the third SQuaRE is probably the most significant of the three meetings. Probably this occurred precisely because of the fact that we had three meetings. i.e. The cumulative effect of such a concentration on the Kadison-Singer Problem for a three year period probably added up to the major breakthrough we made in the third meeting.

I sincerely thank AIM for their continuing support of the Kadison-Singer Problem. Without the chance to concentrate on this problem on a regular basis, the problem may have just died again. I firmly believe we will eventually solve this problem and this would not have been possible without AIM. AIM will be directly responsible for this major advance and is already responsible for several major advances on the problem and on frame theory in general.

13. Modeling cortical spreading depression

May 24 to May 28, 2010
Organized by Robert Miura

Organizer's report:

This AIM SQuaRE is focused on the mathematical modeling of cortical spreading depression (CSD), a slow wave phenomenon in the cortex of different brain structures, which is associated with the spread of depression of the electroencephalogram signal. CSD waves can be instigated in many different experimental animals (Bures et al., 1974). (A recent review of CSD is given in Miura et al., 2007, and two manuscripts in preparation are Huang et al., 2010, and Yao et al., 2010). Discovered by A.A.P. Leão, 1944, these waves are characterized by their slow speeds (1-15 mm/min). While there are many interesting aspects of CSD, its connection to migraine (with aura) in humans makes it a particularly important neurophysiological phenomenon from the clinical point of view (Hadjikhane et al., 2001). The pain-generating mechanism of migraines is not well-understood, and it has been hypothesized that cerebral vasoconstriction following CSD may be partially responsible. In this SQuaRE, our objective was to derive a mathematical model for CSD in which the effects of cerebral blood flow on the CSD activity are taken into account. In future meetings of the SQuaRE, we plan to analyze the model equations and to incorporate more specific experimental data into the model.

AIM provided us with an excellent work environment to conduct our collaborative research, and provided us with continental breakfasts and snacks, and efficient reimbursement for our meals and travel. Unfortunately, three of the scheduled participants, notified us quite late that they were unable to attend the SQuaRE for various reasons, and the time was too short to replace them with any of the alternative participants. We discovered, however, that the remaining five participants worked together very closely and efficiently on the proposed problem.

1. Mathematical Continuum Model

The tissue to be studied is the cortex of a brain structure and is called the gray matter. The gray matter consists of different functional compartments that include the intracellular spaces (ICS) of the neurons and glial cells, and the inner spaces of the vascular tree. These compartments are separate from each other, but all are connected to the extracellular space (ECS) via their membranes. The tissue is formulated as a continuum, following the early work of Tuckwell and Miura, 1978, but includes more realistic mechanisms. Mathematical models of the mechanisms were taken from the works of Kager et al., 2000, 2002, and Bennett et al., 2008. We summarize the various components of the continuum mathematical model below, which includes membrane ionic currents, water movement, and the vascular system.

For the purposes of this report, we limit our discussion to a model of the cortex tissue that is spatially uniform, so the only independent variable is time. The continuum modeling approach assumes that the tissue compartments comprise separate overlapping spaces with exchange of ions through membranes that physically separate the compartments. In general, the CSD phenomenon in the tissue is not spatially uniform, so the effects of spatial diffusion of the ionic concentrations also have to be taken into account. These spatial effects will be included in Brennan et al., 2010.

2. Membrane Ionic Currents

The membrane potentials E_i (relative to the ECS) in each of the membranes connecting the inner compartments to the ECS are governed by the ordinary differential equations

$$C_i \frac{dE_i}{dt} = -I_i \tag{3}$$

where $i = n, g, v$ denote the neuronal, glial, and vascular membranes, respectively. C_i is the membrane capacitance per unit surface area and I_i is the total cross-membrane ionic current per unit surface area due to ionic channels and the pumps and exchangers. Following Kager et al., 2000, 2002, we only consider three major ions, sodium, potassium, and chloride. Thus the ionic channel currents per unit surface area are given as the sum of the sodium currents, the potassium currents, and the leak currents for sodium, potassium, and chloride.

The ion channel currents for Na^+, K^+, and Cl^- are represented by either: 1) the Hodgkin-Huxley model, an electrical circuit analog of the membrane, or 2) the Goldman-Hodgkin-Katz model, an approximate solution to the Nernst-Planck equation, which more precisely describes electro-diffusion through the membrane. Detailed formulas will be given in Brennan et al., 2010.

3. Pumps and Exchangers

Ions also cross the membranes via pumps and exchangers. Included in the membrane models for the neurons, glia, and vessels are the Na-K exchange pump, chloride pump, calcium pump, sodium and calcium exchanger, and potassium chloride transporter. Also included in the membrane model for glial cells is the sodium potassium chloride transporter. The formulas for these currents can be found in Bennett et al., 2008.

Increases in the ICS potassium and ECS sodium concentrations are due to the Na-K exchange pump, which is fueled by the dephosphorylation of ATP. Furthermore, ATP production depends on oxygen that is supplied by the vascular tree, which is included in the model.

4. Volume Changes

One of the characteristics of CSD is that it generates cell swelling, so there is a redistribution of water between the ECS and ICS of the neuronal and glial compartments, but there is no water exchange with the vascular lumen and the tissue. Water channels allow water to cross glial membranes. Although neurons lack these channels, there is water movement through their membranes due to osmotic effects.

As Cl^- is the only extracellular anion in our model, its initial ECS concentration is determined by the sum of both cations: $[Cl]_e = [Na]_e + [K]_e$ to ensure electro-neutrality. We choose the ICS concentration of chloride to achieve isotonicity. Since we only consider Na^+, K^+, and Cl^- ions, the imbalance of ICS ions is offset by immobile ions that cannot cross the membrane.

5. Oxygen Consumption, Vasodilation, and Vasoconstriction

Oxygen is supplied to the tissue through thin blood vessels (capillaries) that are supplied by larger feeder vessels where the oxygen concentration in the blood is fixed. There is an approximately constant pressure drop across the small vessels, which are approximated by cylindrical tubes with uniform radii.

It has been observed that vessels in the vascular tree undergo dilation ahead of the CSD wave and constriction after the wave front passes through. The vessel resistance is a function of effective vessel radius, which is given by an empirical function of the membrane potential. The rate of change in the oxygen concentration depends on the cerebral blood flow rate, which in turn depends on vessel resistance.

Spatial Effects

If spatial effects are included, then almost all of the governing equations incorporate diffusion terms due to membrane potential propagation along membranes and diffusion of ionic species and water. In one-space dimension, we assume there is no diffusion in the ICS of neurons. These additional spatial terms result in a system of diffusion equations with complicated nonlinear source terms. Once the modeling is completed, then the mathematical and computational analysis of this system of equations will be explored.

Acknowledgement

We are grateful to the American Institute of Mathematics for providing us with the opportunity to participate in the SQuaRE program, their hospitality during our stay in Palo Alto, and their generous financial support . We would also like to thank NIH (KCB and JCC), NSERC and MITACS (HH), NSF (RMM), and the Research Grants Council of the Hong Kong Special Administrative Region, China [CityU 102909] (JJW) for providing financial support for our individual research.

References

Bennett, M.R., Farnell, L., and Gibson, W.G., 2008. A quantitative model of cortical spreading depression due to purinergic and gap-junction transmission in astrocyte networks. *Biophys. J.* **95** 5648–5660.

Brennan, K.C., Chang, J.C., Huang, H., Miura, R.M., Wilson, P.L., and Wylie, J.J., 2010. Propagation of cortical spreading depression coupled with vasodilation and vasoconstriction: A continuum model. In preparation.

Bures, J., Buresová, O., and Krivánek, J., 1974. *The Mechanism and Applications of Leáo's Spreading Depression of Electroencephalographic Activity.* Prague, Academia.

Hadjikhane, N., del Rio, M.S., Wu, O., Schwartz, D., Bakker, D., Fischi, B., Kwong, K.K., Cutrer, F.M., Rosen, B.R., Tootell, R.B.H., Sorensen, A.G., and Moskowitz, M.A., 2001. Mechanisms of migraine aura revealed by functional MRI in human visual cortex. *Proc. Natl. Acad. Sci. (USA)* **98**, 4687–4692.

Huang, H., Miura, R.M., and Yao, W., 2010. A simplified neuronal model for the instigation and propagation of cortical spreading depression. In preparation.

Kager, H., Wadman, W.J., and Somjen, G.G., 2000. Simulated seizures and spreading depression in a neuron model incorporating interstitial space and ion concentrations. *J. Neurophysiol.* **84** 495–512.

Kager, H., Wadman, W.J., and Somjen, G.G., 2002. Conditions for the triggering of spreading depression studied with computer simulations. *J. Neurophysiol.* **88** 2700–2712.

Leão, A.A.P., 1944. Spreading depression of activity in the cerebral cortex. *J. Neurophysiol.* **7**, 359–390.

Miura, R.M., Huang, H., and Wylie, J.J., 2007. Cortical spreading depression: An enigma. *Eur. Phys. J. Special Topics* **147**, 287–302.

Tuckwell, H.C. and Miura, R.M., 1978. A mathematical model for spreading cortical depression. *Biophys. J.* **23**, 257–276.

Yao, W., Huang, H., and Miura, R.M., 2010. A continuum neuronal model for the propagation of cortical spreading depression. In preparation.

14. Towards solving Smale's sixth problem

May 24 to May 28, 2010
Organized by Marshall Hampton

Organizer's report:

1. Introduction

The primary goal of our SQuaRE was to make progress towards determining the finiteness of relative equilibria in the Newtonian n-body problem. This was definitely achieved, along with other work on related facets of the n-body problem.

2. Problem efforts and solutions

The following list describes the groups activities during the second half of the SQuaRE. In a previous report I summarized our activities during the first half.

(1) We began our SQuaRE with a report from John Little, who has been thinking about the algebraic geometry of the mutual distance coordinates that most of the participants had been using. He had identified some of the varieties we study in these coordinates as classical structures such as a quadratic Veronese embedding and its secant variety. This was a great example of why the SQuaRE was useful, since John's background in algebra and algebraic geometry gives him a different perspective on the problem. The next day Rick Moeckel provided some further analysis, and he and John continued to work in this area for some of the rest of the SQuaRE.

In particular, as the number of points n considered increases, the variety of the mutual distances of planar configurations becomes more complicated since the number of mutual distances is $n(n-1)/2$ while the dimension of the variety is $2n-3$. Already for $n = 5$, the case of most interest in this SQuaRE, the best way of parameterizing this variety is far from clear. Both during and after the SQuaRE John has been trying to relate his results to a paper by Williams [1] on relative equilibria in the planar five-body problem which combines mutual distances and planar areas as its coordinates.

It would be easy to think that the study of planar point configurations is so old that nothing interesting remains to be found there, but I believe that is definitely not the case. A deeper understanding of that fundamental subject may be one side benefit of this SQuaRE.

(2) Most of the participants spent at least some time thinking about possible counterexamples to the finiteness problem, especially after hearing some of the results of Alain Albouy and Vadim Kaloshin. Gareth Roberts and John Little spent a significant amount of time on this, although to the best of my knowledge did not find any new counterexamples. It seems quite difficult to find these even if non-physical variables (i.e. negative or complex masses and distances) are used.

(3) Vadim Kaloshin sketched some of the methods and results that he and Alain Albouy have obtained for the finiteness problem for the four- and five-body problems. Their approach uses some novel coordinates which many of us spent the rest of the SQuaRE considering. Anders Jensen, in particular, tried to understand how their methods

related to the tropical geometry approach used by Rick Moeckel, Anders, and me in our work.

Vadim and Alain have proved finiteness in the planar four- and five-body problem without needing computer assistance, a remarkable achievement. I am not sure the SQuaRE can take much credit for that, but having Vadim come to explain their work was very important and helpful. Their proof does not cover a few special cases, so some additional work remains to be done. I think all of us spent some time thinking about those special cases.

(4) Anders Jensen and I presented our work on proving finiteness for the spatial central configurations in the five-body problem. Our approach is fairly similar to that used by Rick Moeckel and me in the four-body problem. Similarly to Alain and Vadim's proof, our proof does not cover a short explicit list of special cases given by polynomials in the mass parameters. I spent several days working on finalizing our results with Anders, and trying to eliminate some of our special cases.

One approach Anders and I had considered was to use the Jacobian determinant of a square subset of our equations, since the Jacobian vanishes along curves of solutions. Although hard to compute, these Jacobians did not seem to be helping our analysis much. John Little helped us understand why, and it became clear that adding these determinants will not usually help. This was depressing since we had spent a lot of time on that approach, but at least now we know why it was not helpful.

(5) Gareth Roberts spent a lot of time working on the 'co-circular' configurations. These have some unusual properties in mutual distance coordinates that make them an interesting special case. Gareth was helped by discussions with John Little, Manuele Santoprete, and Rick Moeckel (related as well to the algebraic geometry of planar configurations discussed above).

(6) Gareth and Manuele also worked on studying four-body convex configurations in which there are two adjacent pairs of equal masses. One reason this sort of special case is important is having that pairs of equal masses seems to cause problems in both the planar and spatial five-body finiteness results we have so far.

(7) Eduardo Leandro presented a couple of talks on rational parameterizations of configurations and on degeneracy, bifurcations and enumeration of relative equilibria in the restricted four-body problem (i.e. in which one of the masses is zero). In the second talk, he stressed the applications of Moebius transformations to study the zero-set and determine the sign of a polynomial on specific regions of space. Eduardo also worked on applying Moebius transformations to understand one of the more complicated exceptional mass-polynomials for which finiteness is still open.

This relates indirectly to the work of Alain and Vadim, since their choice of equations involved a lot of variables compared to just using mutual distances. It is somewhat mysterious how our tropical methods relate to the choice of variables, but their approach provided some evidence that it is better to have many simple equations in many variables than a few complicated equations in fewer variables.

3. Conclusion

In the two years since this SQuaRE was proposed we have made a lot of progress on Smale's sixth problem. The five-body case has been almost completely analyzed. The

SQuaRE definitely increased our understanding of the algebraic structure of the Albouy-Chenciner equations. The new methods and equations used by Alain Albouy and Vadim Kaloshin provide much food for thought and hope for further progress.

1Williams, W.L., *Permanent Configurations in the Problem of Five Bodies*, Trans. Amer. Math. Soc. **44** (1938), 562–579.

15. Hausdorff geometry of complex polynomials, positive charge distributions and normal operators

July 12 to July 16, 2010
Organized by Mihai Putinar

Organizer's report:

The four members of the SQUARE group have worked at finalizing the manuscript entitled *"Borcea's variance conjectures"*. The article will be included in a volume dedicated to the memory of Julius Borcea, to be published in 2011 by Birkhäuser, in the series *"Trends in Mathematics"*, with a panoply of very distinguished contributors, and edited by P. Branden, M. Passare and M. Putinar.

Among the new discoveries of the SQUARE group which can be credited to the week spent at AIM are:

1. The discovery of the early geometric insights of **Julius von Sz. Nagy** into the location of critical points of polynomials, cf.:

J. von Sz.-Nagy, *Über geometrische Relationen zwischen den Wurzeln einer algebraischen Gleichung und ihrer Derivierten*, Jahresbericht der Deutschen Math. Vereinigung **27**(1918), 44-48.

J. von Sz.-Nagy, *Über die Lage der Nullstellen der Derivierten eines Polynoms*, Tohoku Math. J. **35**(1932), 126-135.

The group has adapted modern matrix theory techniques and obtained a new proof of a main result of J. Sz. Nagy about circular regions free of critical points centered at a zero of a given polynomial. The same proof applies to some similar classical results of Alexander and Walsh:

J. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. Math. **17**(1915), 12-22.

J. L. Walsh, *On the location of the roots of the Jacobian of two binary forms and of the derivative of a rational function*, Trans. Amer. Math. Soc. **19**(1918), 291-298.

2. A simple example of three unequal point masses in the plane was thoroughly analyzed, with the result of restructuring all weighted variance conjectures of Borcea. As a by-product, a solution to a problem raised a few decades ago by Saff and Twomey was discovered, cf.:

E. B. Saff, J. B. Twomey, *A note on the location of critical points of polynomials*, Proc. Amer. Math. Soc. **27**(1971), 303-308.

3. During the week long meeting a new perspective on Borcea's conjectures was revealed: its natural connection with injectivity criteria for polynomials and rational functions. Although it is too early to assess the importance of this new correspondence, there are a few indications that the large body of known results (classical and modern) referring to univalent analytic functions will lead to major advances in the geometric location of critical points of polynomials.

4. A link between the L^∞-variance appearing in Borcea's conjectures and the notion of Chebyshev diameter (and Chebyshev radius) was established. The latter are well studied concepts in approximation theory and Banach space theory.

In conclusion, the second meeting of the focused research group was overall very efficient and constructive. The four members of the SQUARE have discovered key references and forgotten ideas (some of them almost a century old), they have tuned to the finest detail the forthcoming manuscript and have re-evaluated and better formulated the circulating conjectures arising from Sendov's problem. The matrix theory and univalent function theory approaches (proposed by the group) are totally new and very promising. However, much remains to be done in connection with Sendov's and Borcea's conjectures. The meeting at AIM has managed to put the four participants into a state of obsessive alert in what concerns these questions and the methods developed towards solving them.

16. Nonsmooth and nonconvex optimization problems with matrix variables

July 12 to July 16, 2010
Organized by Jane Ye

Organizer's report:

The traditional continuous optimization theories and techniques are built upon some assumptions such as the smoothness in the objective/constraint functions and certain convexity structure in the constrained set. Recently, originated from many areas of applications such as data mining and statistical learning, new optimization problems with matrix variables and non-smooth constraints have emerged. The non-smooth and non-convex constraints on matrix variables posed a great challenge in the design of reliable and scalable algorithm for this new class of problems. The primary target of our SQuaRE group is to develop effective solvers for this new class of challenging optimization problems.

Since Zhaosong had to leave the SquaRE earlier due to time conflict, our group decided to arrive earlier and make up the lost time. Jane and Zhaosong arrived at the Creekside Inn on Friday afternoon and started to work on the following optimization problem:

$$\text{(P)} \qquad \begin{array}{ll} \min & f(X) \\ \text{s.t.} & \text{rank}(X) \le k-1, \\ & X \in \Omega \subset S_n^+, \end{array}$$

where $k \le n$ is a positive integer and Ω is a subset of S^n_+, the space of all positive semidefinite symmetric matrices. The above generic model covers several well-known hard discrete optimization problems such as data clustering and graph-cut problems, and thus intractable in general. Moreover, since the function rank(X) is integer-valued, the problem is not continuous. On the other hand, we note that in spite of its broad applications, little is known with respect to the above model from an optimization perspective.

To attack the above problem, Jane first suggested to replace the rank constraint rank$(X) \le k-1$ by the constraints $\lambda_k(X) = 0$ where $\lambda_k(X)$ denotes the kth largest eigenvalue of the matrix X. Though $\lambda_k(X)$ is in general not a differentiable function of X, it is Lipschitz continuous. Such a replacement allows us to characterize an optimal solution of (P) by using the generalized gradient associated with certain Lagrange multiplier (the classical gradient does not exist in this case). Then, Jane and Zhaosong discussed the issue of how to develop a numerical procedure to solve such a non-smooth problem. Various approaches such as using a smoothing function to approximate the non-smooth function $\lambda_k(X)$ were considered. After intensive discussion, Zhaosong suggested to write the kth largest eigenvalue as the difference of the sum:

$$\lambda_k(X) = \phi_k(X) - \phi_{k-1}(X).$$

where $\phi_k(X)$ is the sum of the first k largest eigenvalues. Since

$$\phi_k(X) = \max\{\langle X, Y\rangle : 0 \preceq Y \preceq I, \text{trace}(Y) = k\},$$

Zhaosong further suggested to approximate $\phi_k(X)$ via the following problem

$$\phi_k(X,\mu) := \max\{\langle X, Y\rangle - \frac{\mu}{2}\|Y\|_F^2 : 0 \preceq Y \preceq I, \text{trace}(Y) = k\}$$

where $\mu \downarrow 0$ is a smoothing parameter. Once the smoothing function is chosen, our next step is to show that the solution to the following *smooth* optimization problem

$$(\mathrm{P}_\mu) \qquad \begin{array}{ll} \min & f(X) \\ \text{s.t.} & \phi_k(X,\mu) \le \phi_{k-1}(X,\mu), \\ & X \in \Omega \subset S_n^+, \end{array}$$

will converge to the solution of the original *non-smooth* optimization problem (P). Since the problem (P) is non-convex, in general we can only expect to find a stationary point for the problem. Therefore, we plan to prove that the stationary point of the approximating problem (P_μ) converges to the stationary point of the original problem (P) as $\mu \downarrow 0$. In addition, during the discussion Zhaosong also mentioned that it is possible to apply the duality theory for convex programming to study the rank constraint. After some further elaboration, we obtained the following result:

Theorem 4. *The original problem (P) is equivalent to the following problem:*

$$(\mathrm{EP}) \qquad \begin{array}{ll} \min & f(X) \\ s.t. & \mathrm{trace}(S) + zk \le \langle X, Y \rangle, \\ & X - S - zI \succeq 0, S \succeq 0, \\ & \mathrm{trace}(Y) = k-1, 0 \preceq Y \preceq I, \\ & X \in \Omega \subset S_n^+. \end{array}$$

The above result is nontrivial and very interesting. By using more variables and matrix inequalities, we can transform the original non-smooth optimization problem into a smooth optimization problem. Here of course the price to pay is the increased dimensionality. Xiaojun joined the discussion in Saturday afternoon. She pointed out a possibility to use the approaches in her recent works for solving the reformulated smooth problem. On Sunday, Xiaojun, Zhaosong and Jane had more discussions on the details of theoretical proofs and numerical implementation. Jiming arrived on Sunday afternoon and all group members attended a dinner party organized by Professor Yinyu Ye from Stanford University. During the dinner we discussed with Jiming about our progress on rank optimization and he was also excited about the topic since it is closely related to some research projects he has been working on.

On Monday morning, after the introduction meeting to AIM, our group moved to a seminar room to work. Since we have already had an earlier start over the weekend on the rank optimization problem, we all agreed to use this excellent opportunity to identify other research topics that are related to rank minimization and of interests to all of us. Then, every member started to introduce her/his own recent research problems. This is very helpful since some members have not collaborated with each other before. In particular, when Xiaojun reported her recent work on l_p optimization, Jiming noted that some techniques in l_p optimization can help to attack certain hard discrete optimization problems. More specifically, Jiming suggested to consider the following portfolio selection problem:

$$\begin{aligned} \min \quad & x^T Q x \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i = 1, \ \mu^T x = \overline{\mu}, \\ & L y_i \le x_i \le U y_i, \ y_i \in \{0, 1\}. \end{aligned}$$

Here μ_i denotes the expected return of asset i, $\overline{\mu}$ a desirable profit, Q the covariance matrix and x_i the proportion of the total funds invested on asset i. The threshold constraints are necessary for real-world practice which lead to the hard binary constraints defined by

$$y_i = \begin{cases} 1 & \text{if } x_i^* > 0; \\ 0 & \text{if } x_i^* = 0. \end{cases}$$

The binary constraints in the above model makes it extremely hard to solve. As claimed in the literature, problems of size about hundreds variables have not been solved to optimality. Inspired by recent advance in l_p optimization, our group plans to introduce a new continuous optimization approach for the above hard problem.

First step, if necessary, by using a dictionary induced by the vertices from the continuous constrained set, we can remove the constraints $x^T \mu = \overline{\mu}$. Then, instead of the original binary optimization problem, we propose to solve the following problem

$$(\mathrm{QP}_\lambda) \qquad \min \left\{ x^T Q x + \lambda \sum_{i=1}^{n} (x_i^p + (1 - x_i)^p) : \ \sum_{i=1}^{n} x_i = 1, \ x \ge 0 \right\},$$

where p is a number between 0 and 1. The above problem is non-smooth and non-convex due to the use of the l_p norm, $\|x\|^p$ for $p \in (0, 1)$. Moreover, it is not even Lipschitz continuous at the point $x = 0$. After two days of intensive works, by Tuesday we obtained a lower bound result. On Tuesday afternoon, Zhaosong left for another conference and the remaining members continued to work on this problem and by Wednesday we managed to prove the following result:

Theorem 5. *Let x^* be a stationary point of problem* (QP_λ), *then there exist lower and upper bounds, defined only by Q, λ, p such that*

$$1 > x_i^* > 0 \Rightarrow L(Q, \lambda, p) \le x_i^* \le U(Q, \lambda, p).$$

The above result is computationally very appealing since it is the first result showing that a continuous optimization model can ensure that the hard threshold constraints (typically modeled as integer programming) can be satisfied. This provides an novel approach to optimization problems with threshold constraints. On Thursday, Jane and Xiaojun continued to work on the problems and wrote down two technical notes based on our discussions from Saturday to Wednesday, while Jiming went to Stanford to discuss research issues with some colleagues there. Jiming rejoined the group on Friday morning, and we discussed the progress our group made so far and what to do next.

At this point, we believe we had a good start and made substantial progress on the topics during the one-week visit. To work more effectively, we decided to divide the group into two small groups; Xiaojun and Jiming will first focus on the portfolio selection problem while Jane and Zhaosong will continue to work on the rank optimization problem. We will switch the project later depending on the progress on each topic. More specifically, we have also agreed to distribute our tasks as follows before our next group meeting:

1. Jane will coordinate the research activities in our group as well as characterize the optimality conditions for the two problems.
2. Xiaojun will study the smoothing algorithms for the rank optimization and approximation algorithm for the portfolio selection problem.
3. Jiming will focus on the design of effective approximation algorithms for these two problems, extensions and applications.
4. Zhaosong will consider the algorithms and the numerical testing.

In summary, we believe that our SQuaRe group meeting is very productive and fruitful. We plan to continue to work on the two projects and submit the papers to the top journals in the field. We are confident that we can accomplish the goal. Finally, we would like to thank AIM for providing us an excellent opportunity to work together. Our work would not have been possible without the generous support from AIM and its staffs.

10. Minutes of Institute Directors Meeting

11. External Support

12. Funds Received

13. Total Grant Paid Expenses

14. Year 4 Budget

15. Expenses for each Workshop and SQuaRE