LMF: L-functions and Modular forms

Lead PI. John Cremona (Warwick) **Host institutions.** University of Warwick, University of Bristol

Proposal duration. 6 years

Esteem factors and funding track records of team. This is a bi-institutional proposal, shared between the universities of Warwick and Bristol, but with an extensive international network of world class expertise. The core UK team is comprised as follows:

Warwick	Prof John Cremona (**PI**)	Prof Samir Siksek (**co-PI**)
Bristol	Dr Andrew Booker (**PI**)	Prof Brian Conrey (**co-PI**)

All members of this team have extensive experience managing large grants successfully. Cremona and Siksek are PIs for the 2012–2013 Warwick EPSRC Number Theory Symposium (£168,517), and have considerable experience in sponsoring and mentoring postdoctoral research fellows, including two EPSRC PDRAs and six Marie-Curie Fellows since 2006. Siksek is in the fourth year of his EPSRC *Leadership Fellowship* (£544,868) and is the holder of an EPSRC *New Directions for Research Leaders* award (£198,409). Booker is two years into his five year EPSRC *Leadership Fellowship* (£921,570). Conrey is Executive Director of AIM (American Institute of Mathematics) as well as Professor at Bristol, and Booker and Conrey have been PIs on a large-scale project in number theory funded by the NSF (the US analogue of EPSRC), which ran 2008–2011 ($1.1 million).

The UK number theory scene has been transformed over the last decade by the emergence of the groups in Warwick and Bristol. Both now rank among the largest number theory groups in the country, with a continuing influx of young talent, and have established themselves as international leaders in research on L-functions and other areas of number theory. As well as the named PIs and co-PIs, this proposal will directly involve several current members of both groups, including Drs David Loeffler, Bill Hart, Lassina Dembélé and Mehmet Haluk Sengun (Warwick) and Drs Tim Dokchitser, Lynne Walling, Holger Then, Ghaith Hiary, David Platt (Bristol). Hart and Dembélé are EPSRC *Career Acceleration Fellows* (2008-2013 and 2011-2016 respectively); Sengun is a *Marie Curie Intra-European Fellow*; Dokchitser is a *Royal Society University Research Fellow*.

These researchers have been responsible for organising most of the major conferences related to L-functions in the UK over the last few years. Outside the UK, close ties will be maintained and strengthened with, in particular, the University of Washington, University of Waterloo, MIT, and the American Institute of Mathematics.

Vision. Our research vision can be summarised by the following slogan:

Breaking the boundaries of classical L-functions and modular forms, and exploring their applications to 21st-century mathematics, physics and computer science.

L-functions and modular forms are fundamental mathematical objects that encode much of our knowledge of number theory from the last century. They form part of a web of interconnected objects that also include elliptic curves, number fields, and Galois representations. Understanding higher-degree analogues of these objects is a primary challenge of 21st-century mathematics. Our guiding goal is to push forward theoretical and algorithmic developments needed in order to study L-functions and modular forms as thoroughly and systematically as possible. L-functions are to pure mathematics what fundamental particles are to physics: we are compelled to understand the intimate properties of these objects and to understand the decompositions that occur when two L-functions are "smashed" into each other (or "convolved").

Hence, during the course of the project we will develop new theory, create and improve algorithms for computing with L-functions and related objects, discover new properties of these functions, through formulating, testing, and proving fundamental conjectures, and maintain a sophisticated database as a major resource for other researchers worldwide.

Within the overarching theme encapsulated in this vision, we see three interconnected strands on which our research will be focussed:

Higher degree L-functions: classification and computation. Theoretical and computational work on degree 3 and degree 4 L-functions has only started to make significant progress in the past few years. Currently there do not exist efficient methods to work with these L-functions, and rigorous computations with them have not yet been possible, owing to a lack of appropriate algorithms. In fact, there is not even yet an explicit description of all objects which can give rise to degree 3 L-functions. We will address both of these facets: theoretical classification and algorithmic development.

Modularity. L-functions arise from a variety of algebraic, analytic, and geometric objects. We refer to such a relationship as "modularity", reflecting the classical case of degree 2 L-functions associated to GL(2) modular forms. The most famous example of modularity is that of elliptic curves defined over the field of rational numbers, which was established by Wiles and others in the late 20th century, and which has had very far-reaching applications. We will establish theoretical and computational results about a wide variety of arithmetic objects and their relationship to L-functions, taking modularity into the 21st century.

Statistical models. Number theory is deterministic, yet it often is useful to view the objects as random events – a classic example is the distribution of prime numbers. Computational power and fast algorithms have enabled the creation and testing of a wide variety of conjectures about such distributions. Examples of successes are the random matrix conjectures for L-functions. Yet, there are many cases where the data does not support heuristic conjectures, and in some cases the data even appears to contradict proven theorems. The cause of this is that transients in the initial data can persist past the point of current computational power. We will use the wealth of available data, and the significant accumulation of new data created during the course of this project, to develop empirical models and to develop and test new models which combine random matrix theory and other random models with arithmetic effects. The result will be a detailed understanding of the discrepancies between the initial and the limiting behaviour.

Research projects. The award of an EPSRC programme grant, with its emphasis on "flexible mechanisms for providing funding to address significant major research challenges", would provide a considerable step change for the collaboration which has already existed between the PIs and their international partners since 2007, allowing us to vastly expand the scope of the earlier NSF project, with UK-based researchers leading the research forward. Within the framework of developing the online database and encyclopaedia LMFDB ("L-functions and Modular Forms Database", see `http://www.lmfdb.org`) of L-functions, modular forms and related objects, we have identified several research challenges within each of the three strands of our project:

Higher degree L-functions

- L1: establish a theory of rigorous computation of higher-degree L-functions, and develop and implement associated algorithms;
- L2: determine the classification of L-functions of degree ≤ 4, including explicit, proven, examples of every type of object associated to such an L-function.

Modularity

- M1: develop cohomological algorithms to compute Hilbert, Siegel, and Bianchi modular forms;
- M2: elaborate a strategy to prove the (para)-modularity of hyperelliptic curves and abelian surfaces;
- M3: systematically study overconvergent automorphic forms through the development of suitable algorithms;
- M4: devise approaches to studying the p-adic and mod p Langlands correspondences for groups of rank 3 and 4.

Statistical models

Develop theoretical models which explain:

- S1: the preponderance of higher rank elliptic curves in computed data;
- S2: the relative proportion of vanishing of critical values of L-functions in quadratic twist families;
- S3: the distribution of low-lying zeros of L-functions with multiple critical zeros.

With the understanding that has accrued from fundamental lower degree cases, the door is opened for a pioneering effort to tackle the supremely interesting and challenging case of higher degree L-functions. Understanding these is a primary challenge of 21st century mathematics: the world of L-functions opens up in all its glory at this stage! There are at least 10 different categories of L-functions of degree 4, all but a tiny fraction of which have resisted understanding. We are at a point in this project where an intensive effort on the part of the world's experts is needed to ascend to the next level of understanding.

Added value and synergy. This is an inherently large-scale collaborative project, which draws together an array of substantial modern-day challenges under the umbrella of the LMFDB, which will include both a large suite of software developed during the project, and a huge database containing the associated data about L-functions and the related arithmetic and geometric objects. While some of the research goals are independent from each other, meaning that much can be tackled in parallel, the results obtained all feed into the same overriding database. It has been our experience in the past, moreover, that progress on one topic can often inform and enrich progress on completely different topics. This is a particularly appealing feature of the programme grant, which offers the ability to orchestrate fully integrated research on a suite of topics in parallel. A further advantage is the opportunity to plan for the medium- to long-term future, in terms of staffing and infrastructure, combined with the flexibility to respond to new mathematical challenges as they open up.

A key component to our project is to hold regular intensive workshops in Warwick and Bristol, bringing together all the members of both teams and our international collaborators. At these workshops there will be few lectures, allowing substantial time for productive collaborative work as well as reporting, and planning for the next phase of the project. It will be important to be able to plan ahead for these workshops for a number of years, to mark milestones in the project, and not have to rely on piecemeal arrangements.

Computers are to number theorists what colliders are to particle physicists. Aside from their established role as empirical "testers" for conjectures and theories, experiments can often throw up quite unexpected phenomena which go on on to inform (or even reshape) modern theory. In this analogy, the LMFDB is to us what the LHC is to the scientists at CERN. Both are at the threshold of tantalising glimpses into completely uncharted territories (higher degree L-functions for us and the Higgs boson for them). Both would be inconceivable without the large-scale funding needed to sustain and evolve a complex project of this sort (although mathematics is an order of magnitude cheaper).

Following on from the wartime legacy of the likes of Alan Turing, this project represents a significant opportunity to put the UK firmly at the vanguard of a new era of computational number theory. Our project straddles two of the UK's leading centres for computational number theory: the University of Warwick and the University of Bristol. The programme grant will build upon and enhance the current strengths of these groups, but equally, through a suite of national and international conferences it will forge stronger ties with other groups in the UK with suitable expertise. This includes the groups led by Professor Kevin Buzzard at Imperial, Alan Lauder at Oxford, and Yiannis Petridis at UCL.

The goals of this proposal are collaborative by nature. They involve two very rich subjects: L-functions and modular forms. We plan to study many kinds of L-functions associated to classical modular forms, classical Maass forms, Siegel modular forms, Maass forms for GL(3) and GL(4), Hilbert modular forms, Artin L-functions, Hasse-Weil L-functions of algebraic varieties, as well as symmetric power L-functions and Rankin-Selberg convolutions. We will test conjectures that require skill with the analytic, arithmetic, and algebraic aspects of L-functions and modular forms. These include the Generalized Riemann Hypothesis, various probability questions concerning L-functions, the Bloch-Kato conjectures, analytic properties of Hasse-Weil L-functions, the Artin conjecture, and the Selberg eigenvalue conjecture. This project is designed for a group effort and no single individual or collection of isolated individuals is capable of carrying out the fundamental work that we describe. Moreover, this subject area is constantly evolving, with new conjectures and ideas continuously feeding into the area. To help navigate this terrain we propose to set up a highly distinguished scientific steering group, discussed further below.

Timeliness. This is a bold and far-reaching programme of research and implementation, which draws on several fundamental topics that are somewhat under-represented in UK mathematics. Indeed, as recommended in the 2010 *International Review of Mathematical Science* (IRM):

> *There should be more analytic and algebraic expertise in all aspects of modular and automorphic forms, including the Langlands programme. UK research in number theory should look to build in important new directions, rather than reinforcing areas that are already well represented.*

The systematic investigation of L-functions and modular/automorphic forms is a keystone in our proposal and therefore seems perfectly suited to the strategic direction recommended by the 2010 IRM. Furthermore, by building on the momentum of a project originating in North America, we will have firm lines of communication between our UK-based team and numerous internationally world-leading centres for excellence in number theory.

Management. This project is shared between the Universities of Warwick and Bristol, led by the PI at Warwick. As well as day-to-day management of the project's research staff, which will be undertaken by the co-PIs under the direction of the lead PI, there will be a scientific steering group. This group will monitor and help to facilitate our progress, and will also be involved with the direction of the research itself. As well as the four PIs, the steering group will be drawn from an international team of external advisers, which already includes Sir Michael Berry FRS (Bristol, Physics), Kevin Buzzard (Imperial), David Farmer (AIM), Jon Keating FRS (Bristol), Michael Rubinstein (Waterloo), Peter Sarnak (Princeton), Nigel Smart (Bristol, Computer Science and Cryptography), William Stein (Seattle), and Sir Andrew Wiles FRS (Oxford).

Both teams have considerable experience in the management of postdoctoral fellows and research students. Each postdoctoral researcher recruited for the project will have a senior staff mentor who will supervise their work, including forward planning, regular supervisory meetings, review and assistance with preparation of results for publication and wider dissemination. The postdoctoral researchers will play a full part in the research activities in their department, including co-organising research seminars and study groups.

The Mathematics department at Warwick will grant the lead PI a remission from other duties of 50% as a commitment to the smooth running of the project. It is also expected that in both departments, team members would use their DTA funding to recruit PhD students to work on projects connected with our strands of work.

Impact. As well as benefitting researchers in number theory though the LMFDB and associated theoretical advances, we are determined that our project will have a wider impact, both in computer science and physics; to this end we have included both a senior physicist (Berry) and computer scientist (Smart) from Bristol in our advisory group.

Impact within mathematics will come first from the creation of a vast amount of data about a wide range of modular forms and L-functions, which will far surpass in range and depth anything computed before in this area. This in turn will be used to analyse some of the most famous outstanding problems in mathematics, including the Riemann Hypothesis, the Birch–Swinnerton-Dyer conjecture, the Selberg eigenvalue conjecture and the Bloch–Kato conjecture.

Physicists are especially interested in L-functions because they exhibit all of the characteristic features of the spectra of complex quantum systems, yet are very significantly more amenable to numerical computation. They thus provide a source of data that has been of central importance in framing new theoretical approaches to complex quantum systems for over 25 years. The role of random matrix theory in both sets of problems and the similarity between the explicit formula connecting the prime coefficients of an L-function with its zeros and semiclassical trace formulae connecting quantum spectra and classical periodic orbits have been, and continue to be, of fundamental significance. Bristol is widely recognized as being at the forefront in this area. It is also one of the foremost centres in the UK for research in cryptography: number theory and algebra play an increasingly significant role in computing and communications, as evidenced by the striking applications of these subjects to both cryptography and coding theory. For example, the Riemann Hypothesis (one of the Clay Mathematics Million Dollar Millennium Problems) concerns the distribution of prime numbers, and the correctness of the best algorithms for constructing large prime numbers depend on the truth of a generalised version of this 150-year-old unsolved problem. These are algorithms which are used by public-key cryptosystems that everyone who uses the Internet relies on daily, and that under-pin our digital economy.