Preproposal: Computational aspects of the Langlands program

1 Introduction

We propose to advance fundamental research in several mathematical areas brought together by the Langlands program. The conjectures of the Langlands program have served to guide deep research in number theory and related areas; at the same time, these conjectures provide schematic direction to navigate between different subfields of mathematics, linking results in one area by their implications in others. In our proposed ICERM semester, our major activity will be to articulate and explore these connections explicitly, with the specific goal of developing a better understanding of the impact that advances in one area have on those in another. We believe that a semester at ICERM would provide a productive environment for our planned mixture of computation and more traditional pure mathematics research.

There are three specific aspects of the Langlands program which we propose to investigate. The first aspect is to look at both genus 2 curves over $\mathbb{Q}$ and elliptic curves over quadratic fields and to consider their relationship to automorphic forms. This research is modeled on the many successes of the study of elliptic curves over $\mathbb{Q}$ and their relationship to modular forms on GL_2. Here, we will build explicit connections to Siegel, Hilbert, and Bianchi modular forms, and our focus will be on both theoretical and computational work, much of which remains to be done. For example, the precise connection between hyperelliptic curves and Siegel modular forms, the "paramodular conjecture," is very recent, and the generalization of the Shimura-Taniyama conjecture asserting the modularity of elliptic curves over totally real fields is not known in full generality. We also propose to develop new algorithms to compute Maass forms over totally real fields using the Jacquet-Langlands correspondence and working on a (compact) Shimura curve.

The second topic we will consider at ICERM is extensions and refinements of the Langlands program, of which we cite two promising examples. The first example is a "torsion Langlands correspondence": there are Galois groups attached to torsion classes in the cohomology of arithmetic groups that do not come from classical ("liftable") automorphic forms. We believe that these will come to play an important role in the Langlands program, and the precise relationship between them remains rather mysterious. The second example is Sato-Tate groups, which govern the distribution of the Satake parameters of arithmetic objects. For genus 2 hyperelliptic curves over $\mathbb{Q}$ there are 34 such groups and examples are known in each case. The Sato-Tate group of an object is closely related to (and in some sense a refinement of) the L-group of the object. Thus the Sato-Tate group is a potentially valuable ingredient for studying L-functions when one does not have a complete understanding of the automorphic objects that are associated to them, such as those arising from hypergeometric motives.

The third topic is to provide compelling visual and computational displays of the Langlands program in action. Only recently has it become feasible to do large scale verification of the predictions of the Langlands program, to test the conjectures in higher rank cases, and to present the resuls in a way that is accessible widely to mathematicians. The first two activities listed above propose research geared toward advances in computation and algorithms and involving a pool of resources between people in different specialized areas. But it is this third activity that promises the most visibility and consequently is likely to have a long-term impact. Instead of locking away the results of computations in the private computers of individual researchers around the world, results can now be made available on the *L-functions and Modular Forms Database (LMFDB)*, available at `http://www.lmfdb.org/`.

The LMFDB is growing into a comprehensive resource on the areas covered by the Langlands program. It provides both basic and advanced material on a wide variety of mathematical objects and (most significantly) it provides explicitly the connections between those objects. For example, one can move from an elliptic curve, to its L-function, to the associated modular form, to the isogeny class of the curve, and then to the L-function of its symmetric powers. The conections predicted by the Langlands program are, in this light, seen to involve tangible objects which can be studied and understood. We anticipate adding Galois representations and automorphic representations as basic objects in the LMFDB, which will make those complicated objects much more accessible to a wide audience. The results of our proposed research will be incorporated into the LMFDB.

2 Personnel

The organizing committee would consist of Alina Bucur, David Farmer, John Jones, Kiran Kedlaya, Michael Rubinstein, Holly Swisher, and John Voight.

Some of the potential key scientific personnel will be Andrew Booker (University of Bristol), Frank Calegari (Northwestern University), Henri Cohen (Université Bordeaux I), Brian Conrey (AIM), John Cremona (University of Warwick), Henri Darmon (McGill), Chantal David (Concordia University), Noam Elkies (Harvard University), Benedict Gross (Harvard University), Paul Gunnells (University of Massachusetts, Amherst), Ozlem Imamoglu (ETH), Kristin Lauter (Microsoft), Barry Mazur (Harvard University), Kenneth Ribet (University of California at Berkeley) Fernando Rodriguez-Villegas (University of Texas, Austin and ICTP), Peter Sarnak (IAS), Nina Snaith (University of Bristol), William Stein (University of Washington), Michael Stoll (Mathematisches Institut, Universität Bayreuth), Andrew Sutherland (MIT), and Mark Watkins (University of Sydney).

Andrew Booker is a world expert in computing with automorphic forms and L-functions. *Frank Calegari* is interested in the notion of reciprocity linking Galois representations and motives to automorphic forms. *Henri Cohen* is the co-creator of the Cohen-Lenstra heuristics, which predict the structure of class groups of quadratic fields. He led the team that created the PARI/GP computer algebra system. *Brian Conrey* is one of the leading analytic number theorists. His main focus is the analysis of L-functions. *John Cremona* has led the monumental enumeration and tabulation of elliptic curves over $\mathbb{Q}$. By 2012, the Cremona database successfully enumerated all elliptic curves over $\mathbb{Q}$, ordered by conductor, up to the first curve of rank 4 (which has conductor 234,446). This major milestone was reached using improvements in the theory and implementation of algorithms. *Henri Darmon's* research interests include the Hilbert 12th problem and its relation to the Birch-Swinnerton-Dyer conjecture. *Chantal David*, a leading researcher in the area of elliptic curves, has, through her many collaborations, stimulated much computational work on algebraic and arithmetic aspects of L-functions. *Noam Elkies* is one the best and most influential computational and algorithmic number theorists. The Schoof–Elkies–Atkin algorithm (SEA) is an algorithm used for finding the order of or calculating the number of points on an elliptic curve over a finite field. *Dick Gross* is a leading researcher in number theory with interests in connections between automorphic forms and Galois representations, including explicit methods. *Paul Gunnells* has done important work on cohomology of arithmetic groups and automorphic forms. *Ozlem Imamoglu* has gone significant work on classical modular forms, Siegel modular forms, and relationships between them. Her perspective on the connection between these areas will be valuable as we expand these connections to other areas. *Kristin Lauter* is an expert on computational aspects of algebraic varieties, in particular genus 2 curves. *Barry Mazur* is one of the world's leading experts in number theory. His torsion theorem gives a complete list of the possible torsion subgroups of elliptic curves over the rational numbers. Mazur's work on modular curves and his notion of Galois deformations were some of the key ingredients in Wiles's proof of Fermat's last theorem. He is interested in p-adic L-functions and elliptic curves over quadratic fields. *Fernando Rodriguez-Villegas's* work focuses on modular forms, special values of L-functions, the Bloch-Beilinson conjecture and general computational aspects. In recent years his interests expanded to the geometry of character varieties and the moduli space of Higgs bundles. These objects have applications to Langalds duality. With Henri Cohen he has introduced the new area of hypegeometric motives, which is likely to see a blossoming of activity. *Peter Sarnak*, one of the world's top mathematicians, has a unique and broad perspective of mathematics as a whole. He is expert on automorphic forms and L-functions, and has stimulated much work on their algorithmic, numerical, and experimental aspects. *Nina Snaith's* detailed predictions for the moments of L-functions has led to a huge amount of computational work to experimentally test her conjectures.

William Stein is a leading expert in the field of computational arithmetic. He has published numerous papers on the arithmetic of elliptic curves, modular forms and abelian varieties. Stein initiated and is the most prolific contributor to the SAGE project. *Michael Stoll* is an expert both in the theoretic and algorithmic aspects of finding rational points on (elliptic) curves. *Andrew Sutherland* is a rising star of computational number theory. He has done interesting work on hyperelliptic curves and Sato-Tate distributions. *Mark Watkins* is an expert in computation that lies at the interface of analytic, arithmetic, and algebraic number theory.

3 Collaboration at ICERM

The goals of this proposal are by their very nature collaborative. They involve several rich subjects, with L-functions and Galois representations providing the major themes that tie the subjects together. We plan to study many kinds of L-functions including those associated to Artin representations, elliptic curves over number fields, hyperelliptic curves, Siegel and Hilbert modular forms, Maass forms for GL(3) and GL(4), and hypergeometric motives. We will also tabulate Galois representations, and automorphic forms over function fields. Our work will require expertise on analytic, arithmetic, algebraic, and computational aspects of these L-functions and their sources. This project cannot be carried out without a concerted group effort. No single individual or collection of isolated individuals is capable of carrying out such a broad range of work. Furthermore, the LMFDB, which will emphasize how these different points of view and objects interrelate, requires the close collaboration between researchers from different backgrounds.

4 Comparison to previous programs

Our program is unique in a number of ways. It will bring together researchers on computational and algorithmic aspects of automorphic forms, L-functions, and related algebraic and arithmetic objects. It is substantially different from any semester or year long program held in recent years at any of the mathematics institutes:

- Arithmetic Statistics, MSRI, winter 2011. The focus of that program was on limit laws in number theory, such as the work of Manjul Bhargava on counting number fields, or value distributions of arithmetic-geometric objects, with connections to random matrix theory, such as in the work of Keating and Snaith.

- Complex and Arithmetic Dynamics, ICERM winter 2012. We mention this program for completeness because part of the program overlaps with number theory, but there is essentially nothing in common between their methods and their objects of study and our proposed program.

- Automorphic Forms, Combinatorial Representation Theory and Multiple Dirichlet Series, ICERM, winter 2013. The multiple Dirichlet series studied in this program are a branch of number theory that originates with L-functions, but there is essentially no overlap with our proposed program. Specifically, we propose a computational focus, and a study of the close connections between L-functions and other objects. Both of those key aspects are absent from the multiple Dirichlet series program.

- Galois Representations and Automorphic Forms, IAS, 2010-2011. The emphasis of this program was much different than ours. It emphasized the work of Taylor and Wiles on the Shimura Taniyama conjecture, and the Sato-Tate conjecture for elliptic curves and GL(2), all from the deep perspective developed by Taylor and Wiles, with no focus on computation or explicit methods as in our program.