Degenerations in Algebraic Geometry
Problem Session

August 17, 2015

1 Problem 1

(Suggested by Eric Larson) Given the data of a curve X with n marked points lying in projective space $\mathbb{P}^n$, you can forget the curve and remember only the marked points. This can be interpreted as giving a map

$$\mathcal{M}_{g,n}(\mathbb{P}^r, \beta) \xrightarrow{\pi} \mathrm{Conf}(\mathbb{P}^r, n).$$

The basic question I'm interested in is, what can we say about the various possible fiber dimensions or vertical tangent space dimensions?

I have recently finished up a paper that studies when this map is dominant. The upshot is that if $d \geq g + r$, then we can explicitly characterize when the map is dominant.

Remark 1.1. For example, in this range the answer was known by work of Stevens for canonical curves.

To elaborate, consider the case where the map is known to be dominant. What can we say about the fibers of π (e.g. their dimensions, or the dimensions of their tangent spaces)?

Problem 1.2. *Find examples where we can understand behavior distinct from the expected one. There is basically no hope of a general answer, so I mean this as an example-hunting suggestion.*

A specific application would be to extend the range in which we can say when π is dominant. If you have examples where the tangent space dimension is wrong, then you can degenerate to a curve where the tangent space dimension is wrong again.

Remark 1.3. Subject only to the non-negativity of the Brill-Noether number, you can use this result plus a trick of "attaching canonical curves" (where the map π is known to fail to be dominant). When the n points cut out the divisor K_C^2, this fails even more badly, and you can leverage this to extend the range in $\mathbb{P}^3$.

2 Problem 2

(Suggested by Allen Knutson) Let me motivate this by stating a theorem of Brion. Suppose you have a variety X lying in a flag manifold G/P. We say that X is *multiplicity-free* if in cohomology

$$[X] = \sum_{S \subset W/W_P} [X_s],$$

i.e. each zero-dimensional intersection with a general translate of a Schubert variety is empty or 1 point.

Theorem 2.1 (Brion)**.** *If X is multiplicity-free, then X degenerates to the reduced union of Schubert varieties (as predicted by S), and this union is Cohen-Macaulay.*

In particular, if X is reduced and irreducible and multiplicity-free, then X is Cohen-Macaulay.

Problem 2.2. *Classify subsets $S \subset W/W_P$ that deform to irreducibles.*

For example, is $\bigcup_{s \in S} X_s$ Cohen-Macaulay sufficient? The first case to try is $G/P \cong (\mathbb{P}^1)^n$.

Remark 2.3. I have shown that a degeneration of Kazhdan-Lustzig varieties is a union of affine spaces. I think of this as an easier result in that vein.

3 Problem 3

(Suggested by Michel) Let $X \subset \mathbb{P}^2$ be an elliptic curve.

Problem 3.1. *What is the number of degree d rational curves in $\mathbb{P}^2$ meeting X at exactly* 1 *point?*

This is known for $d \leq 8$ by Takashi. Even $d = 9$ would be new and interesting.

The motivation is from local mirror symmetry.

One approach is to degenerate X to 3 lines. Also, is a tropical approach possible?

4 Problem 4

(Suggested by Brian Osserman) Consider forming a nodal curve X by gluing together two rational curves at $g + 1$ generic points. Is X Brill-Noether general?

Problem 4.1. *Does X have a $g^r_{d_1,d_2}$ if and only if $\rho(g, r, d_1 + d_2) \geq 0$ (subject to some condition on d being "relatively balanced")?*

Caporaso ("Brill-Noether theory of binary curves") thought about this. She proved some results for $r \leq 2$, using a definition of "relatively balanced" which is the weakest reasonable one. Another possible definition could be to require $0 \leq d_i \leq g - 1$.

Caporaso's motivation is to give an alternate proof of the Brill-Noether theorem by degeneration to such curves, which would also give a new criterion for Brill-Noether generality. For me, the motivation is related to my definition of limit linear series for curves of non-compact type. The answer to this question has concrete consequences on the dimension of limit linear series for pseudocompact curves.

It is possible that the answers are distinct depending on the definition of "relatively balanced." That would be interesting.

Remark 4.2. This is closely related to Problem 1.

5 Problem 5

(Suggested by Brian Harbourne) The *Bounded Negativity Conjecture* says that for X a surface which is either rational or over characteristic 0, there exists n_X such that $C^2 \geq n_X$ for all reduced $C \subset X$.

Nobody knows how to approach this. Recently people have started looking at a variant called "h-constants." Let C be a singular reduced plane curve of degree d. Look at

$$h(C) := \frac{d^2 - \sum_{x \in C} m_x^2}{\#\text{singular points}},$$

where m_x is the multiplicity of the singular point x. (This is essentially the self-intersection of the proper transform in the blowup.)

Problem 5.1. *Is h_C bounded below uniformly?*

If this is true, then the Bounded Negativity Conjecture holds for rational surfaces in characteristic 0. (In characteristic $p > 0$, h is not bounded below. One could ask what the interesting statement would be in characteristic p.)

It is known that if C is a union of lines, then $h(C) > -4$, while the best achieved example is something like -3.36. The point is to find a configuration of lines with many high-multiplicity intersections (basically, the point is that you want to avoid simple crossings).

Remark 5.2. This is only interesting in characteristic 0, because over a finite field you can easily find C sending $h(C) \to -\infty$ by taking *all* (rational) lines.

Problem 5.3. *For irreducible curves, the best known examples have $h \to -2$. Can we do better?*

Remark 5.4. Roulleau has produced configurations of cubics with $h \to -4$.

Remark 5.5. Over $\mathbb{R}$, we know that $h \to -3$ and $h \geq -3$. Over $\mathbb{Q}$, no analogous result is known.

One could go further and ask if the Bounded Negativity Conjecture is true for tropical surfaces.

6 Problem 6

(Suggested by Ravi Vakil) Here is an idea for a tropical approach to interpolation of points in $\mathbb{P}^2$. We want to show that given d and $m_1, \ldots, m_s$, if the expected dimension is positive then the interpolation problem has the right number of solutions (this is the *SHGH Conjecture*).

By adding conditions, we may assume that the expected dimension is 1. We can try to do this by find a single isolated (i.e. non-deforming) and liftable tropical solution.

7 Problem 7

(Suggested by Omid Amini) Here is a "weak version" of Nagata's conjecture.

Problem 7.1. *When Nagata predicts $h^0 = 0$, show that $dL - m_1E_1 - \ldots - m_sE_s$ is not big by using "convex methods."*

Remark 7.2. This might actually imply the strong form of Nagata's conjecture.

8 Problem 8

(Suggested by David Jensen) The *Maximal Rank Conjecture* predicts the Hilbert fuction of a general curve (for fixed g, r, d such that $\rho(g, r, d) \geq 0$ and $r \geq 3$).

The Hilbert function measures the rank of the map $\operatorname{Sym}^m H^0(X, L) \to H^0(X, L^{\otimes m})$. The Maximal Rank Conjecture predicts that this has maximal rank for general X and L. Concretely, this means that it is either injective or surjective.

We proposed a combinatorial approach via the notion of *tropical independence*. However, we can only solve the combinatorial problem for $h(2)$ (the case of quadrics). The next step would be $h(3)$, i.e. cubics.

Problem 8.1. *Prove this for all m.*

The current argument is sort of naïve. We make a choice of functions in $\operatorname{Sym}^m H^0(X, L)$ which we want to be independent. Thus, the argument works more naturally in the injective case because there is no choice needed. In that case, you have a bunch of functions and you want to show that $\binom{m}{r}$ "m-wise" sums are independent. To do that in the case $m = 2$, this becomes a combinatorial game.

After this, one coud ask about higher syzygies (viewing the Hilbert function as describing the first term of the syzygy).

Problem 8.2. *Use degeneration techniques (tropical independence or limit linear series or both) to understand higher syzygies.*

This is known by Green's conjectures for generic curves under the canonical embedding, by non-tropical techniques.

Problem 8.3. *Can one give a new proof of Green's theorem by tropical methods?*

9 Problem 9

(Suggested by Erwan) This is a question about Welschinger invariants of real rational surfaces, which is a real analogue of genus 0 Gromov-Witten invariants.

Specifically, we are interested in a signed count of real curves through d_1 real points and d_2 pairs of complex conjugate points (such that $d_1 + 2g_2 = 3d - 1$) in $\mathbb{P}^2$. (The sign is something like (-1) raised to the number of isolated nodes.)

Remark 9.1. For $\mathbb{P}^2$ there are several formulas in terms of d_1, d_2. However, these are so opaque that even the sign of the total sum is unclear.

For some specific cases, the sign can be deduced by degenerating $\mathbb{P}^2$.

- If $d_1 \approx 0$ or $d_2 \approx 0$, then the sign is known.
- If $d_2 = 0$, the sign is always non-negative.
- If $d_1 = 0$ or 1 (i.e. minimal depending on the parity), then the sign is $(-1)^{(d-1)(d-2)/2}$.

Problem 9.2. *What about when $d_1 = 2$ or 3? (Conjecturally, it is $(-1)^{(d-1)(d-2)/2+1}$ for $d \gg 0$.)*

Remark 9.3. If we consider the same problem on a more general real surface X, then the signs tell us something about the topology of the real algebraic surface X.

There are two possible approaches to this question.

1. Analyze the combinatorics of the known formulas. However, this might be less interesting.
2. Find some nice degeneration that makes the answer clear. For instance, all the curves might have the same sign, or there could be a natural injection from curves of one sign into curves of the other sign.

For example, for the minimal d_2 or d_1 case the sign is deduced by such a degeneration.

What shape to the formulas have?

- One formula is in terms of Floor diagrams.
- Another formula is in terms of open WDVV equations.

10 Problem 10

(Suggested by Jake) Consider a line bundle L on $\mathbb{P}^1$. Suppose we want to study the Grassmannian of $g^r_d \subset \Gamma(\mathbb{P}^1, L)$. We can consider conditions on g^r_d's, e.g. as follows. Pick a point $x \in \mathbb{P}^1$, and considering the vanishing flag at x. This defines a complete flag, by stratifying by the degree of vanishing at x. You can then impose Schubert conditions on the linear series (which concretely means imposing conditions on the vanishing orders).

These Schubert conditions are indexed by partitions. More generally, we can do this at several points $x_1, \dots, x_s$ with several partitions $\lambda_1, \dots, \lambda_s$. Let S be the set of g^r_d satisfying all these conditions, i.e. the intersection of all the Schubert conditions.

Theorem 10.1 (Eisenbud-Harris '80s)**.** *We have* $\operatorname{codim} S = \sum_i |\lambda_i|$.

This has an interesting relation to real algebraic geometry.

Theorem 10.2 (Muken-Tarasov-Varchenko)**.** *If all the x_i are real, and* $\sum_i |\lambda_i| = \dim\{g^r_d\}$*, then S consists of real reduced points.*

Now suppose that S is a curve rather than a finite set.

Theorem 10.3. *If all x_i are real and* $\sum_i |\lambda_i| = \dim\{g^r_d\} - 1$*, then* $S(\mathbb{R}) \subset S^{smooth}$.

Problem 10.4. *Is S smooth? (i.e. are there singular points over $\mathbb{C}$?)*

There doesn't seem to be any reason that this should be true, but in the (admittedly few) examples that I've explored, I've found no singular examples.

An idea for an approach is to use tautological bundles on $\overline{M_{0,s}}$ to describe equations characterizing the locus in $M_{0,s}$ where S is singular (and perhaps show that this is away from the points where all the x_i are real).

Example 10.5. I've already checked the simplest case, e.g. $G(2, 5)$ with the partitions $s = 5$ points all being a single box.

The next simplest cases would be with longer partitions, or $G(2, 6)$.

In this case, you can write the equations for the singular locus as a sum of squares (an obvious way to prove that there are no real solutions). In the example I looked at, the squares had the form $(x_i - x_j)^2$, which suggests a boundary geometric interpretation (points colliding).

B. Osserman suggests:

Problem 10.6. *Reprove Mukhin-Tarasov-Varchenko theorem using degenerations. Extend it to cominiscule G/P.*