Lemma. 0.1 *For $r \geq 1$, as $n \to \infty$ we have*

$$\frac{\pi(p_n^r)}{p_n^r} \sim \frac{e^\gamma}{r} \prod_{k=1}^{n} \Big(1 - \frac{1}{p_k}\Big) \tag{1}$$

Proof. By the PNT we have

$$\frac{\pi(p_n^r)}{p_n^r} \sim \frac{1}{\log(p_n^r)} = \frac{1}{r \log(p_n)}, \tag{2}$$

and by Mertens' result [give a reference to a modern textbook, not the original] we have

$$\frac{1}{\log(p_n)} \sim e^\gamma \prod_{k=1}^{n} \Big(1 - \frac{1}{p_k}\Big). \tag{3}$$

Combining (2) and (3) gives (1).