Interpolation on $\mathbb{P}^2$ via degenerations

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Motivating Questions

K an algebraically closed field, $\mathrm{char}(K)$ arbitrary for now.

$S = \{p_1, \ldots, p_s\} \subset \mathbb{P}^2$ distinct points.

$m_1, \ldots, m_s \geq 0$ desired orders of vanishing.

The Fundamental Questions
(a) What is the least degree among homogeneous polynomials

$$0 \neq F \in K[x, y, z] = K[\mathbb{P}^2] \text{ with } \mathrm{ord}_{p_i}(F) \geq m_i \text{ for all } i?$$

(b) For each t what is the vector space dimension spanned by all such F with $\deg(F) = t$?

Note: the answer to FQ (b) also answers FQ (a).

Algebraic Perspective

$Z = m_1p_1 + \cdots + m_sp_s \subset \mathbb{P}^2$ denotes subscheme defined by

$$I(Z) = I(p_1)^{m_1} \cap \cdots \cap I(p_s)^{m_s} \subset K[\mathbb{P}^2].$$

$I(mZ) = (I(Z))^{(m)} = I(p_1)^{mm_1} \cap \cdots \cap I(p_s)^{mm_s} = m^{th}$ symbolic power of $I(Z)$

Note: $I(Z) = I(Z)_0 \oplus I(Z)_1 \oplus I(Z)_2 \oplus \cdots$ is a homogeneous ideal.

The Hilbert function of $I(Z)$ is $h_{I(Z)}(t) = \dim_K(I(Z)_t)$.

Thus $h_{I(Z)}(t)$ answers FQ (b).

Geometric Perspective

Let $\pi_S : X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at the points of S.

Let $E_i = \pi_S^{-1}(p_i)$ and let L be the pullback of a line.

Let $F_Z(t)$ denote $tL - m_1E_1 - \cdots - m_sE_s$.

Write $h^0(F_Z(t))$ or $h^0(X, F_Z(t))$ for $h^0(X, \mathcal{O}_X(F_Z(t)))$.

Fact: $$h^0(X, F_Z(t)) = h_{I(Z)}(t)$$

(i.e., answering FQ (b) same as computing h^0 of line bundles on X)

Focus on general points: Nagata's Conjecture

Question: What is the value of $h^0(X, F_Z(t))$?

Note: $h^0(X, F_Z(t))$ depends on the points p_i.

Assume the points p_i are general; then $h^0(X, F_Z(t))$ depends only on t and the m_i.

For $s \leq 9$, the answer is due to Castelnuovo
[Ricerche generali sopra i sistemi lineari di curve piane,
Mem. Accad. Sci. Torino, II 42 (1891)].

Nagata's Conjecture (1959): Assume $s \geq 10$ general points. Then

$$h^0(X, F_Z(t)) = 0 \quad \text{for} \quad t \leq \frac{m_1 + \cdots + m_s}{\sqrt{s}}.$$

SHGH Conjecture

SHGH Conjecture: a conjecture for $h^0(X, F_Z(t))$ (p_i general, any s)
Segre (1961)-Harbourne (1986)-Gimigliano (1987)-Hirschowitz (1989)

It's not obvious from Segre's version how to get an explicit value for $h^0(X, F)$. His version of the conjecture is: Let F be a divisor on a blow up X of $\mathbb{P}^2$ at general points. If $h^0(X, F)h^1(X, F) > 0$, then $|F|$ has a nonreduced fixed component.

My version is: Let X be the blow up of $\mathbb{P}^2$ at generic points. Then (a) any reduced irreducible curve C with $C^2 < 0$ must have $C^2 = -1$ and be smooth and rational, and (b) $h^0(X, P)h^1(X, P) = 0$ for every nef divisor P.

This does give an explicit value for $h^0(X, F)$ for any divisor F, since if (a) is true, using the Weyl group action on divisor classes, it's not hard to either show F is not effective or to produce an integral Zariski decomposition $F = P + N$, where P is the nef part, in which case one has $h^0(X, F) = h^0(X, P)$.

BNC

One of the things that make proving SHGH hard is that the points are general or generic. It would be nice to formulate a version of SHGH that applies to a blow up of $\mathbb{P}^2$ at any points, or even a version that applies to any smooth projective surface, and which gives SHGH in the case of blow ups of $\mathbb{P}^2$ at generic points. No one has done that for part (b), but there is such a thing, more or less, for part (a): The Bounded Negativity Conjecture (BNC).

BNC has an oral tradition that goes back at least to Enriques. It says that for each smooth complex projective surface X there is a bound b_X such that $C^2 \geq b_X$ for all effective reduced (or equivalently, reduced and irreducible) divisors C on X. (If we restrict X to rational surfaces, we can drop the requirement that X be a complex surface. This is because the only surfaces known without bounded negativity are in characteristic $p > 0$ but are not rational.) The SHGH Conjecture implies that $b_X = -1$ for blow ups of $\mathbb{P}^2$ at generic points.

References for the SHGH Conjecture

B. Segre, Alcune questioni su insiemi finiti di punti in geometria algebrica, Atti Convegno Intern. di Geom. Alg. di Torino, (1961), 15–33.

B. Harbourne, The geometry of rational surfaces and Hilbert functions of points in the plane, Proceedings of the 1984 Vancouver Conference in Algebraic Geometry, CMS Conf. Proc., 6 (1986) 95–111.

A. Gimigliano, On linear systems of plane curves, Thesis, Queens University, Kingston (1987).

A. Hirschowitz, Une conjecture pour la cohomologie des diviseurs sur les surfaces rationelles génériques, J. Reine Angew. Math., 397 (1989), 208–213.)

Special case of SHGH Conjecture

For simplicity, assume $s \geq 10$ general points,
$m_1 = \cdots = m_s = m \geq 0$, and let $E = E_1 + \cdots + E_s$.

The following is a special case of the SHGH Conjecture:

Main Conjecture Today (MCT):
For $t \geq 0$ (and $s \geq 10$ general points),

$$h^0(tL - mE) = \max\left\{\binom{t+2}{2} - s\binom{m+1}{2}, 0\right\}$$

(equivalently, $h^0(tL - mE)h^1(tL - mE) = 0$)

Exercise:
(a) MCT implies Nagata's Conjecture.
(b) For $t \leq \sqrt{s}m$, Nagata's Conjecture implies MCT.

Degeneration Techniques for verifying MCT

Main idea: specialize somehow and use semicontinuity (i.e., cohomology cannot decrease under specializations in a flat family).

Four main techniques:

1. Specialize the points to a curve and recursively residuate.
2. Specialize to infinitely near points and unload.
3. Multistep specializations (Horace's method).
4. Degenerate the plane

Consider the problem of computing $h^0(X, F)$ for $F = 10L - 2E$ for $s = 21$ general points.

SHGH predicts $h^0(F) = 3$ and $h^1(F) = 0$.

Method 1: Specialize the points to a curve

Specialize the $s = 21$ points to a smooth quartic curve C'.

Let C be the proper transform of C' to the blow up X' of the specialized points.

The hope is that we can choose C' so that $h^1(X', F) = 0$. We have the following exact sequences:

$$0 \to \mathcal{O}_{X'}(F - C = 6L - E) \to \mathcal{O}_{X'}(F) \to \mathcal{O}_C(F) \to 0$$

$$0 \to \mathcal{O}_{X'}(F - 2C = 2L) \to \mathcal{O}_{X'}(F - C) \to \mathcal{O}_C(F - C) \to 0$$

To know $h^1(X', F) = 0$ (and hence $h^1(X, F) = 0$), it's enough to know $h^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) = 0$ and $h^1(C, F) = h^1(C, F - C) = 0$.

What happens:

$$\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{O}_{X'}(F - C = 6L - E) & \rightarrow & \mathcal{O}_{X'}(F) & \rightarrow & \mathcal{O}_C(F) & \rightarrow & 0 \\
 & & \text{so } h^1 = 0 & & & & \text{but} & & \\
 & & & & & & \deg(F|_C) = -2 & & \\
 & & & & & & \text{so } h^1 > 0 & &
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{O}_{X'}(F - 2C = 2L) & \rightarrow & \mathcal{O}_{X'}(F - C) & \rightarrow & \mathcal{O}_C(F - C) & \rightarrow & 0 \\
 & & h^1(X', 2L) = & & & & h^1 = 0 & & \\
 & & h^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) = 0 & & \text{so } h^1 = 0 & & & & \\
 & & & & \text{here} & & & &
\end{array}$$

This specialization is too special since $h^1(X', F) = h^1(C, F) > 0$.

Method 2: Specialize less but “unload”

Specializing all of the points to a quartic is too special; suppose you specialize only, say 19, of the points to the quartic C'.

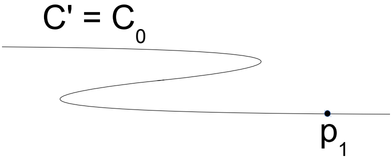
$$0 \to \mathcal{O}_{X'}\begin{pmatrix} F - C = \\ 6L - E_1 - \cdots - E_{19} - 2E_{20} - 2E_{21} \end{pmatrix} \to \mathcal{O}_{X'}(F) \to \mathcal{O}_C(F) \to 0$$

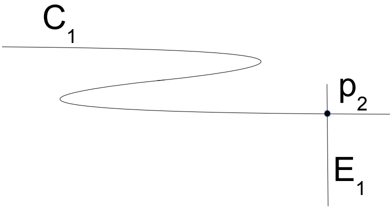
$$0 \to \mathcal{O}_{X'}\begin{pmatrix} F - 2C = \\ 2L - 2E_{20} - 2E_{21} \end{pmatrix} \to \mathcal{O}_{X'}(F - C) \to \mathcal{O}_C(F - C) \to 0$$

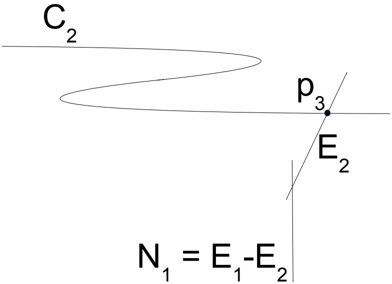
The method fails, since $h^1(X', 2L - 2E_{20} - 2E_{21}) > 0$.

We need to be do something with the last two points, so we take p_1 to be a general point of C', p_i, $i = 2, \ldots, 19$, successive points of C' infinitely near to p_1, then p_{20} and p_{21} infinitely near to p_{19} but not near C'. So $C = 4L - E_1 - \cdots - E_{19}$. Let C_i be the proper transform after blowing up i of the points, so $C' = C_0$.

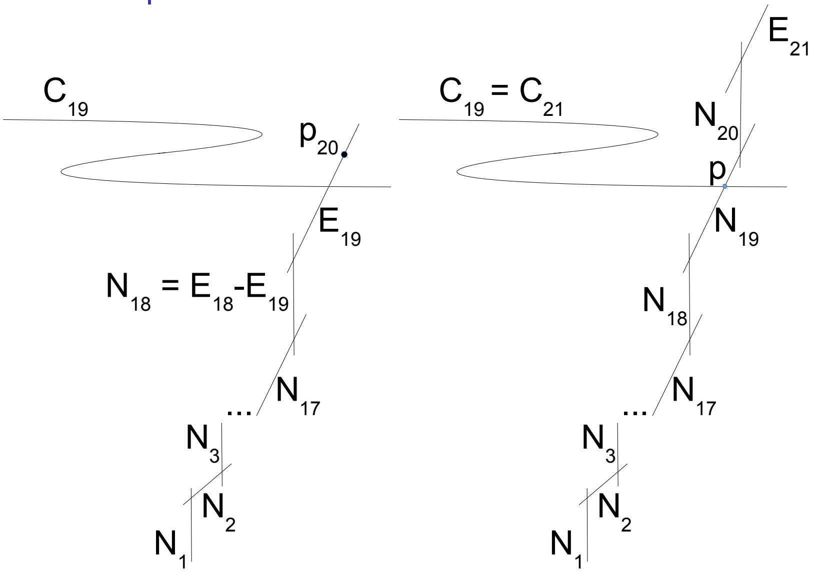
The blow ups







More blow ups



The exact sequences

$$
\begin{array}{ccccccccc}
0 & \to & \mathcal{O}_{X'}(F-C) & \to & \mathcal{O}_{X'}(F) & \to & \mathcal{O}_C(10L|_C - 38p) & \to & 0 \\
 & & & & & & h^1 = 0 & & \\
 & & & & & & & & \\
0 & \to & \mathcal{O}_{X'}(2L - 2E_{20} - 2E_{21}) & \to & \mathcal{O}_{X'}(F-C) & \to & \mathcal{O}_C(L|_C - 19p) & \to & 0 \\
 & & h^1 = 2 & & & & h^1 = 0 & &
\end{array}
$$

This almost worked. We get $h^1 = 0$ on the right using the special form of the divisor. To get $h^1 = 0$ on the left we use “unloading” (i.e., restrict to various N_i in such a way as to preserve cohomology).

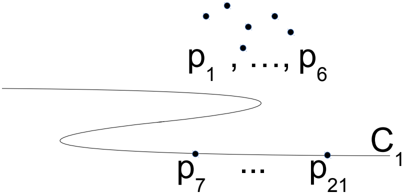
$$
\begin{array}{ccccccccc}
0 & \to & \mathcal{O}_{X'}(F-C) & \to & \mathcal{O}_{X'}(F) & \to & \underset{h^1=0}{\mathcal{O}_C(10L|_C-38p)} & \to & 0\\
0 & \to & \mathcal{O}_{X'}(6L-E-E_{19}-E_{21}) & \to & \mathcal{O}_{X'}(6L-E-E_{20}-E_{21}) & \to & \underset{h^0=h^1=0}{\mathcal{O}_{N_{19}}(-1)} & \to & 0\\
0 & \to & \mathcal{O}_{X'}(6L-E-E_{18}-E_{21}) & \to & \mathcal{O}_{X'}(6L-E-E_{19}-E_{21}) & \to & \underset{h^0=h^1=0}{\mathcal{O}_{N_{18}}(-1)} & \to & 0\\
 & & & & \cdots\ \text{(unloading)} & & & & \\
0 & \to & \mathcal{O}_{X'}(6L-E-E_{1}-E_{21}) & \to & \mathcal{O}_{X'}(6L-E-E_{2}-E_{21}) & \to & \underset{h^0=h^1=0}{\mathcal{O}_{N_{1}}(-1)} & \to & 0\\
 & & & & \cdots\ \text{(unloading)} & & & & \\
0 & \to & \mathcal{O}_{X'}(6L-E-E_{1}-E_{2}) & \to & \mathcal{O}_{X'}(6L-E-E_{1}-E_{3}) & \to & \underset{h^0=h^1=0}{\mathcal{O}_{N_{2}}(-1)} & \to & 0\\
0 & \to & \mathcal{O}_{X'}(2L-E_1-E_2-E_{20}-E_{21}) & \to & \mathcal{O}_{X'}(6L-E-E_{1}-E_{2}) & \to & \underset{h^1=0}{\mathcal{O}_C(6L|_C-21p)} & \to & 0\\
 & & & & \cdots\ \text{(unloading)} & & & & \\
0 & \to & \underset{h^1=0}{\mathcal{O}_{X'}(2L-E_1-E_2-E_3-E_4)} & \to & \mathcal{O}_{X'}(6L-E_1-E_2-E_3-E_5) & \to & \underset{h^0=h^1=0}{\mathcal{O}_{N_{4}}(-1)} & \to & 0
\end{array}
$$

(In positive characteristics one must be careful about the choice of C'; in characteristic 0 there is no problem.) For details, see Theorem 1.3 of Harbourne and Roé, Linear systems with multiple base points in $\mathbb{P}^2$, Adv. Geom. 1 (2003), 41–59.

Method 3: Multistep Specializations (Hirschowitz's méthode d'Horace)

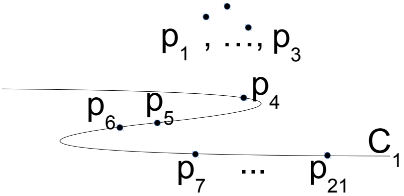
Aside: "Horace" refers to a French legend, the point of which was to obtain success by a divide and conquer strategy. Anyway, again we compute $h^1(10L - 2E) = 0$ for $s = 21$ and again we do not specialize all of the points to a single curve.

Step 1: Specialize the last 15 points to general points on a smooth cubic, so $C_1 = 3L - E_7 - \cdots - E_{21}$:



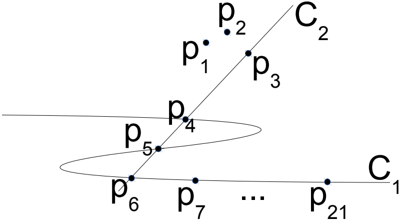
$$0 \to \mathcal{O}_{X_1}(7 - 2_6 - 1_{15}) \to \mathcal{O}_{X_1}(10 - 2_{21}) \to \underset{h^1 = 0}{\mathcal{O}_{C_1}(\text{degree} = 0)} \to 0$$

Step 2: Specialize three more of the points to the cubic



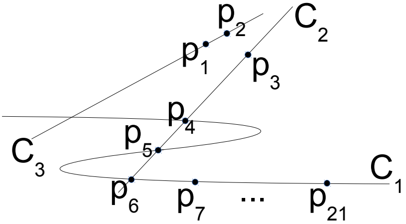
$$0 \to \mathcal{O}_{X_2}(4-2_3-1_3) \to \mathcal{O}_{X_2}(7-2_6-1_{15}) \to \underset{h^1=0}{\mathcal{O}_{C_1}(\text{degree}=0)} \to 0$$

Step 3: Specialize three points on the cubic to a line
$C_2 = L - E_3 - E_4 - E_5 - E_6$ through one of the free points:



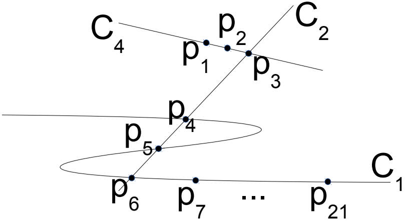
$$0 \to \mathcal{O}_{X_3}(3-2_2-1) \to \mathcal{O}_{X_3}(4-2_3-1_3) \to \underset{h^1=0}{\mathcal{O}_{C_2}(\text{degree}=-1)} \to 0$$

Step 4: Restrict to the line through the two double points, so $C_3 = L - E_1 - E_2$:



$$0 \rightarrow \mathcal{O}_{X_3}(2 - 1_2 - 1) \rightarrow \mathcal{O}_{X_3}(3 - 2_2 - 1) \rightarrow \underset{h^1 = 0}{\mathcal{O}_{C_3}(\text{degree} = -1)} \rightarrow 0$$

Step 5: Specialize the three points to a line $C_4 = L - E_1 - E_2 - E_3$:

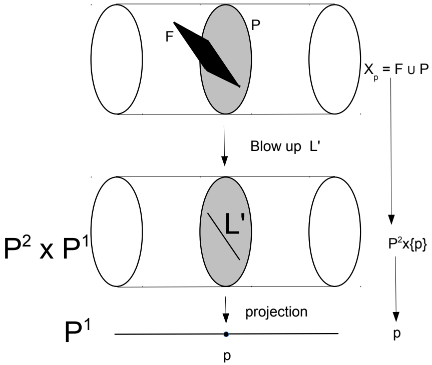


$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_{X_4}(L) & \to & \mathcal{O}_{X_4}(2 - 1_2 - 1) & \to & \mathcal{O}_{C_3}(\text{degree} = -1) & \to & 0 \\ & & h^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) = 0 & & & & h^1 = 0 & & \end{array}$$

For more details see A. Hirschowitz, La méthode d'Horace pour l'interpolation à plusiers variables, Manuscripta Math., vol. 50 (1985), 337–388.

Degenerate X (Ciliberto-Miranda)

Assume $K = \mathbb{C}$.



$$\begin{tikzcd}
F \arrow[r, phantom, "\subset"] \arrow[d] & X_p = F \cup P \arrow[r, phantom, "\subset"] \arrow[d] & \mathrm{Bl}_{L'}(\mathbb{P}^2 \times \mathbb{P}^1) \arrow[d] \\
L' = L' \times \{p\} \arrow[r, phantom, "\subset"] & \mathbb{P}^2 \times \{p\} \arrow[r, phantom, "\subset"] & \mathbb{P}^2 \times \mathbb{P}^1
\end{tikzcd}$$

F is abstractly the blow up of $\mathbb{P}^2$ at one point, with exceptional divisor $E = F \cap P \cong L'$, and P is $\mathbb{P}^2$.

Degenerate X (Ciliberto-Miranda)

A linear system $\mathcal{L}$ given by dL through s general points of multiplicity m on a general fiber $\mathbb{P}^2 \times \{t\}$ can be chosen to degenerate to a linear system $\mathcal{L}_p$ on X_p where $\mathcal{L}_p|_P$ is given by $(d-k)L$ through $s-b$ general points of multiplicity m on P and to and $\mathcal{L}_p|_F$ is given by $dL-(d-k)E$ through b general points of multiplicity m on F.

To compute $h^i(X_p, \mathcal{L}_p)$ we need to compute the restrictions to the components, taking into account matching conditions along L'. The matching conditions turn out to behave fairly nicely, so one gets an inductive procedure which often allows a successful conclusion.

For details see C. Ciliberto and R. Miranda, Degenerations of planar linear systems, Crelle, 501, (1998) 191–220. For a detailed worked out example, also see arXiv: 0910.1171 by Ciliberto, Dumitrescu, Miranda and Roé.

Toric Degeneration method

There is also a toric degeneration approach. For this see arXiv: 1104.1755.pdf by Dumitrescu.