Getting Started with PROBLEM SOLVING:
A Trio of Friendly Problems

By Mary Fay-Zenk
Editing assistance from Tom Davis, Katy Early, and Jim Tanton
Graphics by Tom Davis

Overview

The three problems presented for this extended lesson have both individual and cluster appeal. ***The Handshake Problem*** is a natural opener for the beginning of the school year, but could easily be presented to students at any time. It's a situation-friendly and simple question: *Suppose you walk down to the corner some afternoon and there are six of your friends standing around. How many handshakes would there be if each person shakes each and every other person's hand once?* The second problem, ***All Possible Diagonals***, asks students to draw all possible diagonals in eight regular polygons, record their findings in a table, and use this information to generalize the number of diagonals in an *n*-gon. The third challenge, entitled ***Triangular Numbers***, asks students to generate a rule for finding the *n*th triangular number.

Since each of these problems can be visualized or acted out quite readily, the problems can be accessible at some level for virtually every middle school student. In addition, each of these problems can be solved using a recursive pattern, which means that students who are organized and persevering can complete a great deal of the task and feel very successful. Ultimately, of course, one goal is that students would learn to recognize patterns that can be generalized. Even better than this, however, is to have students begin to explain *why* the patterns hold true and how the situation might change if one or more of the parameters were to vary. Throughout the lessons, encourage students to move from "what is happening" questions to "why are they happening" and "what if" questions.

By offering all three of the problems to your students over a period of time, there is the opportunity to draw them into the problem-solving experience at a deeper level and allow students to practice and strengthen their ability to generalize and recognize underlying themes/parallels within various contexts.

Problem I: ***The Handshake Problem***

Learning Objectives: By the end of this lesson students will be able to

- Organize data obtained by thinking through a tangible experience
- Abstract the essentials of a mathematical analysis from a real-life situation
- See patterns in the numbers generated by shaking hands with an increasing number of persons

Materials Required:

No specific materials are needed to present this problem other than paper and pencil for small groups of students to record their work and a whiteboard or overhead projector for the teacher to record and systematize the thinking of the class.

Instructional Plan:

This lesson is primarily an interactive, teacher-directed activity from which students will generate number patterns to analyze.

1. Present students with this simple scenario.

Suppose you walk down to the corner of your street some afternoon and there are six of your friends standing around. How many handshakes would there be if everyone shook hands with each other person exactly once? (Shaking hands with yourself is not allowed!)

Make certain students understand the situation clearly and assign them to groups of three or four to begin to act out/talk out the scenario and collect data. Encourage students to make a table recording the information they agree upon so they can share with the class after a few minutes.

> Although students may be tempted to join immediately with another group in order to have a group of seven to act out the situation, persuade them to start with their smaller group of three or four students and try to find a way to find a pattern that would lead them to a solution they can defend. If students are insistent about tackling the larger problem, you might promise that they can try to act out the group of seven-at-the-corner after some initial pattern searching has been accomplished. This will serve two purposes: first, it could be a lot of fun to do all that handshaking (which is pretty important for middle school students in math class!); secondly, it will demonstrate the power of "trying a simpler case" since there are so many handshakes and so many ways to count them that accurate data collection will require an organized approach or be quite prone to error.

2. Monitor how the groups are working. If you think the small groups are not making progress on the task, remind them to act out simple cases. How many handshakes would happen if three people gathered? How many handshakes if four people are at the corner?

After you see some progress has been made, ask groups to report back to the class how many handshakes seven friends would generate and to be prepared to describe the method they used to reach their conclusion. If, as you might expect, their answers vary, take time to look at several simpler cases that can be fairly easily verified with students actually shaking hands and build a table.

3. Model organizing data:

Number of people	Number of handshakes
1	**0**
2	**1**
3	
4	
5	
6	
7	
8	

Create a systematic list with the number of people and the number of handshakes carefully labeled so that students can refer to the data to look for patterns. Most students will see a recursive pattern emerging: each time another person joins the group the number increases by the previous number of people. If, for example, four people are already gathered (and, therefore, six handshakes have happened) and a fifth person joins the cluster, that new person will have four people to greet, adding four new handshakes for a total of 10!

Spend some class time drawing out the various ways students were able to think about the problem and honor their methods. See if you or they can connect the thinking of one group with that of another. Any sketches or representations students can show to concretize their thinking will undoubtedly help other students understand and develop further useful approaches.

Questions for Students:

1. How many people are needed for a single handshake?

2. How would you draw a picture or a diagram that shows all the possible handshakes between four people?

3. Suppose a fifth person comes along to this group of four. How many additional handshakes will there be?

4. Will there always be (n-1) new handshakes if an nth person joins a group of (n-1) people? Why or why not?

Assessment Options:

Oftentimes, the process of problem solving is engaging enough in itself so that it doesn't require a separate assessment to ascertain that thinking has occurred. However, it can be very useful to both teachers and students to have students "revisit" a classic question in some way, perhaps as a bonus problem on a routine test or quiz. For example, as a follow-up to ***The Handshake Problem***, a teacher might ask students, *How many handshakes will happen if six different students each shake hands exactly once?* As part of their response, students should be expected to use a table or a diagram to communicate their thinking and validate their conclusions.

Students who are able to show their reasoning and correctly give the answer will demonstrate that they have internalized the process and truly learned from the class experience. Using a smaller number (6 rather than 7) in the bonus problem will allow students to feel more confident that they can solve the problem independently in a test or quiz situation. Students who are not able to re-solve the problem will benefit from seeing how other students approached the situation once again when the question is reviewed. By making this a bonus question on a test or a quiz, and not penalizing those who are not able to reconstruct the solution, you as the teacher are making it clear that you value the work they did, want them to be able to re-visit their thinking, and you were never expecting them to "memorize" the answer. This is a most important stance to take if we truly want students to be comfortable and courageous problem solvers.

Extensions:

1. Have students continue the pattern they discovered in their tables, and see if they can predict what number of handshakes would happen if there were ten or twelve friends gathered at the corner.

2. Encourage students to develop a symbolic system for representing the number of persons and the number of handshakes. A couple of ways that might be good options include using dots for persons and lines for handshakes or using letter names for each person, i.e., A, B, C, and segment notation, AB, BC, CA, for handshakes. (This may facilitate identifying generalized patterns for the problem since the students can then see the problem at a glance.)

3. IF they seem ready, challenge students to find a formula that will generalize the number of handshakes generated by any number of persons.

Teacher Reflection:

As you begin presenting interesting problems to students, be keenly aware of their readiness and practice patience! The most important outcome is that students deeply understand the processes involved and can internalize the fundamental mathematical principles that are being applied. Sometimes a principle is as simple as recognizing that any one of the persons can start the handshaking; there is nothing inherently unique about person A, for example. Another critical understanding is that every handshake is a "shared" experience. If you count the handshake of AB and AC, you can't count the handshakes BA and CA unless you remove the duplications subsequently. Conversely, you can count all the persons' handshakes as long as you do compensate by dividing by two.

Although it is this realization that leads most directly to the generalized formula for the handshake problem, if this solution doesn't come from the students themselves, be patient! Wait until after the students work on the ***All Possible Diagonals*** challenge to discuss this approach.

Solutions for ***The Handshake Problem***

3. Model organizing data:

Number of people	Number of handshakes
1	**0**
2	**1**
3	**3**
4	**6**
5	**10**
6	**15**
7	**21**
8	**28**

Questions for Students

1. How many people are needed for a single handshake?

It takes two people to have a handshake.

2. How would you draw a picture or a diagram that shows all the possible handshakes between four people?

You might use stick figures, capital letters, or points to represent the four persons and lines between them to represent the arms extended in handshakes.

3. Suppose a fifth person comes along to this group of four. How many additional handshakes will there be?

If a fifth person comes along to a group of four, there will be exactly four people to shake hands with. He or she can't shake hands with him or herself!

4. Will there always be (n-1) new handshakes if an nth person joins a group of (n-1) people? Why?

There will always be (n-1) new handshakes when a next new person (n) joins a group, since that person shakes hands with all the persons previously in the group and not with him or herself.

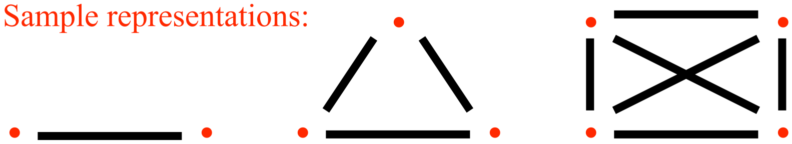
Extensions:

1. Have students continue the pattern they discovered in their tables, and see if they can predict what number of handshakes would happen if there were ten or twelve friends gathered at the corner.

If there are ten friends gathered, there will be a total of 45 handshakes. If there are twelve friends, there will be 66 handshakes.

2. Encourage students to develop a symbolic system for representing the number of persons and the number of handshakes. A couple of ways that might be good options include using dots for persons and lines for handshakes or using letter names for each person, i.e., A, B, C, and segment notation, AB, BC, CA, for handshakes. (This may facilitate identifying generalized patterns for the problem since the students can then see the problem at a glance.)

Sample representations:



2 people, 1 handshake   3 people, 3 handshakes   4 people, 6 handshakes …

OR

Persons: A, B	Handshake: AB
Person: A, B, C	Handshakes: AB, AC, BC
Persons: A, B, C, D	Handshakes: AB, AC, AD, BC, BD, CD, and so on

3. IF they seem ready, challenge students to find a formula that will generalize the number of handshakes generated by any number of persons.

The number of handshakes with n people is: $\boldsymbol{n\,(n\text{-}1)\,/\,2}$

This formula can be thought of as multiplying the number of people (n) times the number of handshakes each of the person makes (n-1) and dividing by 2, which eliminates the double counting since every handshake requires two participants.

Problem II: ***All Possible Diagonals***

The beauty of this lesson is that it is so simple. Almost universally, drawing the diagonals captivates students. The patterning that takes place as students move from polygon to polygon has a calming effect. Most students will take a systematic approach working from one vertex to all other possible diagonal vertices and know intuitively there is a pattern here to be discovered.

Learning Objectives: By the end of this lesson students will be able to

- Abstract the essentials of a mathematical analysis using regular polygons
- See patterns in the numbers and in the geometric designs generated by connecting all possible diagonals in polygons with an increasing number of sides
- Recognize how important precision is in helping to bring patterns to light
- Learn what it means to generalize for an unknown number in a given situation

Materials Required:

Student worksheet (have plenty of extras on hand for students who need to start over)
Pencil, straight edge, patience and precision

Instructional Plan:

1. Begin the lesson by distributing the *All Possible Diagonals* worksheet. Review the instructions on the sheet with the students and make sure they understand what a diagonal is. Make certain each student has a straight edge for drawing the diagonals. Let students know that it will really enhance their work if they can be very neat when drawing the diagonals for each polygon. They should use sharpened pencils to make their line segments for each diagonal just as precise as possible.

> Depending upon how well developed a student's fine motor control is, he or she may have more or less difficulty completing the first part of this particular task. Many students become fascinated with the beautiful, symmetric patterns that appear when all the diagonals of the polygons are drawn, but the patterns will not be so beautiful if the lines are not drawn precisely and aligned properly with the vertices. The symmetry that emerges from a well-drawn product also helps students know whether or not all the diagonals have been connected. Allow students to have a "practice" sheet if they need one and/or to have a new sheet after they have developed some facility drawing the diagonals. (Attention to detail is a valuable skill to develop in any discipline and one worth encouraging.)

2. Monitor to see how comfortably students are able to proceed. Enjoy these moments of quiet and concentration.

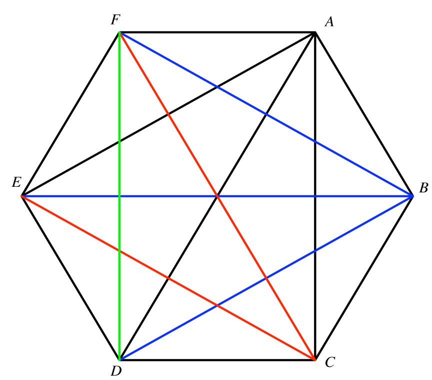
Encourage students to watch for patterns in the designs and in preparation for filling out the table. Of course, if they wish, students may fill out the table as they complete the diagonals for each polygon.

As you notice students beginning to finish up, encourage them to share the table entries with nearby neighbors and see if they agree. Ask them to work in small groups to uncover patterns in the table so their observations can be shared with the class.

3. Using student input, build a table that duplicates the one the students have on their paper for the class to view and discuss the patterns they see. Try not to reveal or suggest any comparisons to ***The Handshake Problem*** yourself. Allow one of the students to recognize that the column for the "Number of Sides and Diagonals" contains almost the same list as the "How many handshakes?" problem.

4. Develop several visual schemes for seeing patterns and several numerical methods of counting the diagonals. For example, consider the hexagon and the following counting schemes.

One systematic way to count the number of diagonals for the hexagon is to count all the diagonals from point A, add the new diagonals from points B, C, and so on. This reflects the recursive formula usually seen first by the students.



3 + 3 + 2 + 1 for a total of 9 diagonals

Looking at the diagram a bit differently, you can see that there are 3 diagonals from any vertex and 6 vertices. Multiply 3 x 6 to get 18 diagonals. Next divide by 2 since each diagonal has been counted twice, once from each of its endpoints. That leaves 9 distinct diagonals!

The second approach begins by choosing any vertex of the hexagon and noticing how many diagonals it has. Multiply this number of diagonals by the number of vertices and you will have twice as many diagonals as there really are because each diagonal shares two vertices. Just divide by two and your answer is precisely correct!

This view of diagonal counting can be generalized to most students' satisfaction. In the case of the hexagon, there are six sides and six vertices. In fact, in every polygon, the number of sides and the number of vertices is the same; so that we can state a rule for any polygon, let's call this number *n.* Ask students to check their drawings and their tables to see if they can find a pattern. The number of diagonals from each vertex of *every* polygon appears to be (n-3). Therefore, to find the number of diagonals, we multiply n (n-3) and divide by 2.

Questions for Students:

1. Looking at your drawings on the worksheet as a reference, describe why the number of diagonals appears to be (n -3) in the case of each polygon shown. Will that always be the case? Explain.

2. What would be a general rule, using words or a formula, for the number of sides *plus* the number of diagonals for an n-gon?

Assessment Options:

If you feel there is a need for separate assessment as part of this exploration, you might allow the students to choose to respond to either the **Questions for Students** above or to explore the **Extensions** given below. Whichever question(s) the students choose allows you to evaluate their understanding and be assured that they are internalizing key vocabulary and fundamental concepts.

Giving the students a choice about which question(s) to focus on for assessment both respects their sense of their own understandings and provides a level of ownership for the task they decide to do. Although questions 1 and 2 are somewhat straightforward, students may be uncomfortable because they involve unknowns. Extension number two is probably the most challenging of the inquiries but it is very accessible since it allows students to combine their thinking with drawings and use a simple organized method for collecting information no matter their previous experience with formal problem solving.

Extensions:

1. Each of the polygon shapes that were given on the worksheet was a regular polygon, which means that the length of each side and the angle measure at each vertex were the same. Does the number of possible diagonals change if you have a non-regular polygon? Can you explain why or why not?

2. The diagonals of a regular polygon divide the shape into a variety of non-overlapping regions. If the polygon is not regular, do the diagonals divide the polygon into the same number of regions? Explore with the polygons at least through the regular and non-regular heptagon before coming to a conclusion.

Teacher Reflection:

Patience and perseverance, good questioning strategies, allowing students time, small group interaction, analysis of different approaches—this problem encourages a teacher to apply all of these good classroom practices. Remember the goal: to teach problem solving, which is a way of thinking, a habit of mind, a willingness to tackle a question, take on a challenge. George Polya, often dubbed the "Father of Problem Solving," famously said, "It is better to solve one problem five ways than to solve five different problems."

Consider whether or not you were patient with students' readiness and progress. Did you honor the progress they made and create an environment that made them feel comfortable sharing their work? Were students able to persevere with the drawing precisely, with collecting data, and/or with searching for patterns? Did you take enough class time to draw out all the various approaches students tried? Did you record the ways different students developed their lists? See if your students can come up with five different ways to approach this problem!

If you have enough time, allow small groups of students to record their approaches on large sheets of paper; this becomes a wonderful class record and, again, validates their work. Posting these sheets around the classroom offers yet another way to encourage further thinking and explorations.

For personal reflection: These two problems demonstrate a key role that definitions and limitations play in actively doing math. To solve the handshake problem, it needs to be completely clear that a person does not shake hands with him or herself, just as a single point does not determine a line. It is also central to the situation to agree that a handshake between Alice and Bob is not different from the handshake between Bob and Alice. Similarly, in our definition of diagonals, the line segment AB is the same as the line segment BA. Though these considerations may seem so "obvious" to us as teachers that they do not seem significant, they are crucial understandings and should not be glossed over.

There is also a fundamental parallel between these two problems, ***The Handshake Problem*** and ***All Possible Diagonals***, that transcends the problems themselves, which can be a true "aha!" moment. At their root, the two situations are the same. The persons are the vertices; the handshakes are the diagonals and vice versa. Problems such as these, which transcend the details, often play a significant role in proving some of the classic mathematical challenges. Here is an example of abstract reasoning that students can experience and truly understand early on in their mathematical education. Problems such as these, which have the same underlying structure, are called "isomorphic."

Solutions for ***All Possible Diagonals*** Problem

Questions for Students:

1. Looking at your drawings on the worksheet as a reference, describe why the number of diagonals appears to be (n -3) in the case of each polygon shown. Will that always be the case? Explain.

A diagonal is defined as a line segment that connects two nonconsecutive (nonadjacent) vertices of a polygon. Thus, by definition, three points will always be excluded from forming a diagonal with a specific vertex: the point itself and the two points which are adjacent (and form sides of the polygon when connected). Every other vertex does define a diagonal for the polygon. Hence, in every case, (n-3) will be the number of diagonals that can be drawn from a given vertex in a polygon.

2. What would be a general rule, using words or a formula, for the number of sides plus diagonals for an n-gon?

The general rule or formula is: $n\,(n\text{-}1)/2$

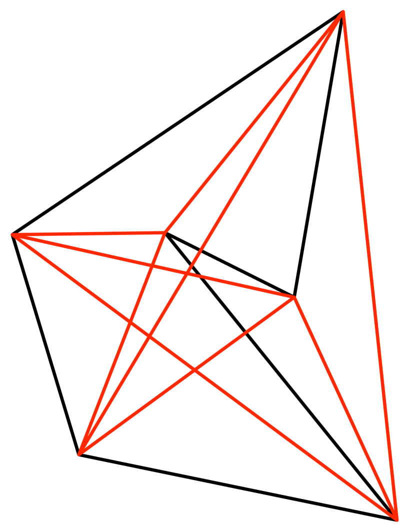
If both the sides and the diagonals of a given polygon are being counted, then the number of segments, which should be counted from each vertex, will be (n-1). The number of vertices will still be n. When we multiply n x (n-1), we will have “double counted” each segment, and therefore we need to divide by two.

Extensions:

1. Each of the polygon shapes that were given on the worksheet was a regular polygon, which means that the length of each side and the angle measure at each vertex were the same. Does the number of possible diagonals change if you have a non-regular polygon? Explain why or why not.

A given polygon is determined by the number of vertices and the number of sides that it has (both being the same). Thus, since the number of potential diagonal endpoints and the number of vertices remain the same, the same rule, $n\,(n\text{-}3)/2$, would still give the number of diagonals for that polygon whether it is regular or non-regular, convex or concave.

Please note that a convex polygon is one similar to the regular polygons on the worksheet in that all the diagonals fall completely within the boundary of the polygon itself. A concave polygon is one whose vertices and sides are arranged so that at least one of the diagonals lies outside the polygon’s boundary as in the following sample:



In this sample concave polygon, notice that the sides are drawn in black and the diagonals in red.

2. The diagonals of a regular polygon divide the shape into a variety of non-overlapping regions. If the polygon is not regular, do the diagonals divide the polygon into the same number of regions? Explore with the polygons at least through the regular and non-regular heptagon before coming to a conclusion.

The number of non-overlapping regions created by regular and non-regular polygons of the same order does vary in some cases. In fact, in a regular hexagon the number of regions created by the diagonals and the side boundaries of the hexagon is 24, but in a non-regular hexagon, the number of regions increases to 25. The regular and non-regular heptagons both have the same number of interior regions: 50. However, the regular and non-regular octagons have different numbers of interior regions. The regular octagon has 80 regions and the non-regular octagon has up to 91. Encourage the students to look back at their *All Possible Diagonals* worksheet to see if they can determine why these regular and non-regular polygons have different numbers of interior regions when the other polygons have not.

Referring to the regular hexagon on the worksheet, you can see there is a center point at which three diagonals meet. In a non-regular hexagon, there is not the same symmetry and so the intersection of these three diagonals forms a triangle (and hence one more region), not a point! The regular octagon has 8 such points where three diagonals meet at one point and one place where four diagonals intersect. In a non-regular octagon, some, possibly all, of the "collapsed" regions are visible, adding up to 11 more interior regions to the octagon.

There is a general formula for the relationship between the number of diagonals, the number of interior points and the number of regions in any (i.e., a non-regular) polygon. The formula is: $d + p + 1 = r$.

Problem III: ***Triangular Numbers***

The third in this series of problems presents the challenge of discovering a rule for the *n*th triangular number in a more conventional way, but still with a visual component, the triangular arrays. This allows students who are more comfortable with a visual focus to get started on the problem and to begin to generate the table required. After using the illustrations of the triangular number arrays to confirm the number of dots in the first few triangles, students have the option of using a pattern to generate the Total (*T*) column or they may continue to count the dots through the eighth figure. Whichever method makes more sense to students, the outcome should be the same: the new number of dots in the triangular arrays will be the previous number plus the next counting number.

Learning Objectives: By the end of this lesson students will be able to

- Abstract the essentials of a mathematical analysis from an initial illustration of dots in a specific formation
- Continue the pattern of triangular numbers using a recursive process or visualizations
- Use patterns and previous experiences to help formulate a rule for the *n*th element of the set of triangular numbers
- Practice what it means to generalize for an unknown number in a given situation

Materials Required:

Student worksheet (***Triangular Numbers***), pencil, available scratch paper; for Extension 2, students will need a copy of the resource sheet, ***Figurative Number Arrays***.

Instructional Plan:

1. Begin the lesson by distributing the ***Triangular Numbers*** worksheet.

Since students have already experienced working with the class on ***The Handshake Problem*** and on their worksheets with the ***All Possible Diagonals*** exploration, they should be comfortable with seeing a similar table along with the triangular number arrays.

Review the directions with the class asking them to go as far as they can in responding to the different prompts on the sheet. Most students will be able to complete the list for the total number of dots through the ninth triangular number.

2. Monitor individual progress and, when students appear to have had enough individual time, encourage small group work and discussion.

For the questions asking students to predict how many dots are in the 50th and 100th triangular numbers, some persistent students may be willing to use a recursive pattern to obtain the 50th triangular number, and they should applauded for their tenacity. But finding the 100^{th} triangular number can be even more rewarding when it has been found by searching the table for patterns that can be generalized.

Encourage students to look back through the table of entries 1 through 9 and see if they can find some patterns that would lead them to an answer in the *T*-column. Allow students to work in small groups to do a "group-think." Ask students to report back to the class when they believe they have found a useful pattern.

3. Draw out student thinking, taking time to share the patterns that different students notice and asking students to reflect on how they found any patterns they identified.

Many students' first reactions will be, "I don't know" or "I just thought of it," but pushing them to reflect back on what was going through their minds when they observed the pattern can help them understand a process they want to claim and, perhaps, to be able to generalize later on.

4. If /when a student or a small group comes up with the rule for the nth triangular number, ask the student(s) to describe how they thought about the generalization. The more input, the better. There are several ways that the formula can be discovered and every way is valuable for the student(s) to articulate and for their classmates to hear.

Some students may have previously seen Gauss' Method of Summation and know the formula, $\frac{1}{2} n (n + 1)$, which will give the result for the *n*th triangular number as well. Emphasize that a close look at the arrangement of the dots in the equilateral triangular arrays will verify that we are doing, in fact, is adding all consecutive numbers to *n*.

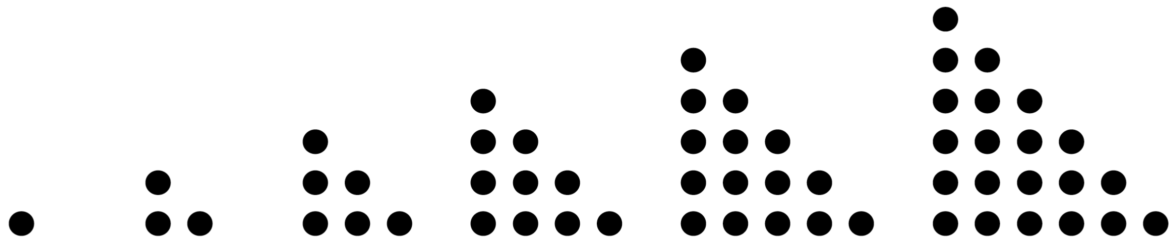
If students or small groups do not find a rule for the nth triangular number, don't panic! Encourage them to think back to ***The Handshake Problem*** or ***All Possible Diagonals*** and reexamine some of the leads they followed to develop the formula $\frac{1}{2} n (n-1)$. See if re-visiting the earlier problem allows them to reinvigorate their thinking.

Questions/Prompts for Students:

1. Have you tried looking at the simpler cases to see if you could find a convenient connection between numbers in the n-column and numbers in the T-column that could establish a pattern?

2. Did you look for a multiplicative relationship between the number of the figure, *n*, and the number of total dots, *T*, in the array for that figure? Alternatively, you might look at T/n and see if you can find a pattern that allows you to establish a general pattern.

3. A right triangle arrangement of dots can also display the pattern of triangular numbers conveniently:



n=1, T=1 *n=2, T=3* *n=3, T=6* *n=4, T=10* *n=5, T=15* *n=6, T=21*

Does this particular display help you think of any additional patterns regarding triangular numbers?

4. Can you apply any of the techniques used in solving ***The Handshake Problem*** or ***All Possible Diagonals*** to this problem? What are some common elements among the three problems?

Assessment Options:
If students have not yet seen Gaussian summation, you might use the following as your next bonus problem since one of the classic solutions to the Gaussian sum involves "double counting" the sum and dividing by two to get the actual answer much as we can double count the triangular numbers in an oblong array (question #3 above).

> Sum the numbers, 1 + 2 + 3 + 4 + 5 + … + 9 + 10;
> 1 + 2 + 3 + 4 + 5 + … + 29 + 30; and/or
> 1 + 2 + 3 + 4 + 5 + … + 99 + 100,
> showing your thinking as well as your answer.
>
> You will receive *double* bonus points if you can generalize the summation of the numbers from 1 to *n*!

Alternately, you might use one of the Extensions for an assessment question.

Extensions:

1. Find a general formula for the sum the first *n* odd numbers. (If you need a hint to get started, make a table and find the sums of the first several odd numbers and look for a pattern.)

2. Explore other figurative numbers that can be represented geometrically and whose *n*th term can be generalized. You might start with *Square Numbers*, which can be represented by dots in a square arrangement or array, and *Oblong Numbers*, which can be represented in a rectangular array having one dimension one unit longer than the other.

Even more challenging would be *Pentagonal Numbers* and *Hexagonal Numbers*, which can be represented by dots in a pentagonal array or hexagonal array respectively.
In order to consider the *Pentagonal Numbers* and *Hexagonal Numbers*, use the displays from the ***Figurative Number Arrays*** resource page.

Teacher Reflection:

Looking back on the experiences you and your students have had in trying to solve these three friendly problems can be an excellent opportunity to reflect upon the process and art of problem solving itself. What encouraged students? What interested them and caused conversation in small groups? What led to good whole class discussions? Did the diagrams and visualizations assist students in understanding the problem clearly?

How able were the students to make the connections between problems, especially between ***The Handshake Problem*** and the ***All Possible Diagonals*** challenge?

For yourself as a teacher, how comfortable were you in allowing students to struggle with finding a pattern? Did you find yourself wanting to step in with your ideas, your way of seeing the solution? Were you able to find the right questions to prompt further thinking among the students?

Thinking about thinking and the processes involved in developing strategies to solve non-routine problems can really expand the intellectual prowess of your students and it can give you as the teacher many insights into how students learn. This "meta-cognition" is one of four key principles Polya considered central to problem solving, and he claimed that much can be gained by taking the time to reflect and look back at what you have done, what worked and what didn't. Sometimes in lesson planning this is termed "closure," and it is a valuable component of effective lesson design.

Solutions for the **Triangular Numbers Problem**

Questions for Students:

1. Have you tried looking at the simpler cases to see if you could find a convenient connection between numbers in the n-column and numbers in the T-column that could establish a pattern?

Some of the patterns that students find are:

You add the next value of n to the previous total of T to obtain the new T.

You add consecutively larger values to T to get the new T.

2. Did you look for a multiplicative relationship between the number of the figure, n, and the number of total dots, T, in the array for that figure? Alternatively, you might look at T/n and see if you can find a pattern that allows you to establish a general pattern.

n	*Some multiplicative patterns …*		T
1		1 x 2/2	1
2		2 x 3/2	3
3	Students might see that 3 times two is 6.	3 x 4/2	6
4		4 x 5/2	10
5	Students might see that 5 times three is 15.	5 x 6/2	15
6		6 x 7/2	21
7	Students might see that 28 is four times 7.	7 x 8/2	28
8		8 x 9/2	36
9 …	Students might see that 45 is five times 9.	9 x 10/2	45
30 …		30 x 31/2	465
100		100 x 101/2	5050
n		n x $(n + 1)/2$	$\frac{1}{2} n\ (n+1)$

Once students notice the patterns in red in the chart, students may also see the pattern between these whole number products (in blue). Ten is two and a half times 4; 21is three and a half times 6. The numbers in blue in the chart above extend the pattern even more elegantly so that the formula $\frac{n \quad (n+1)}{2}$ emerges.

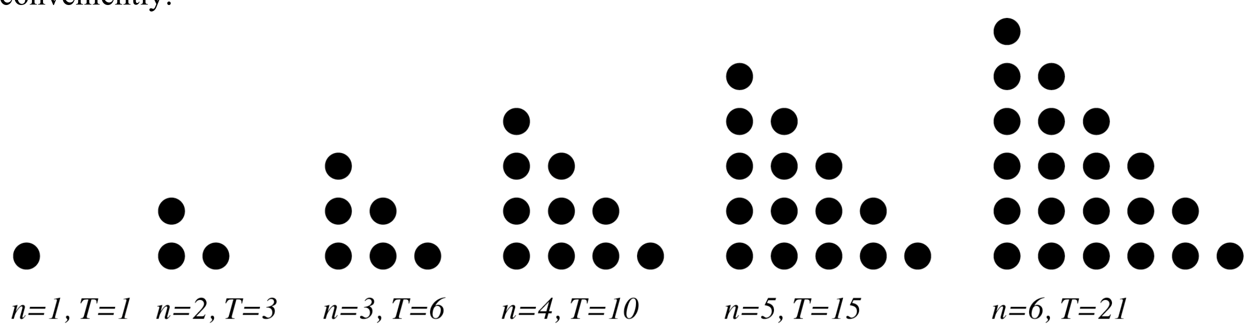
Another way of using multiplication: Looking at the n-column, you can obtain T by multiplying two consecutive n values and dividing by 2. For example, 4 x 5 = 20 and 20 divided by 2 gives 10, which is the total number of dots in the fourth triangular number; similarly, 5 x 6 / 2 = 15. Of course, this gives the same results as the blue pattern above, but it is discovered differently, using the *n*-column to obtain the two numbers and noticing they are two times as large as what you have in the *T*-column.

Still other students may approach the problem by dividing *T* by *n*, as in the table below.

T	n	*Some patterns seen by division: T/n ...*	*T/n*
1	1	1/1	1 or 1
3	2	3/2	3/2 or 1.5
6	3	6/3	2
10	4	10/4	5/2 or 2.5
15	5	15/5	3
21	6	21/6	7/2 or 3.5
28	7	28/7	4
36	8	36/8	9/2 or 4.5
45	9 …	45/9 …	5 …
465	30 …	465/30 …	31/2 or 15.5 …
5050	100	5050/100	101/2 or 50.5
$\frac{1}{2}n(n+1)$	n	$\frac{1}{2}n(n+1)/n$	$\frac{1}{2}(n+1)$

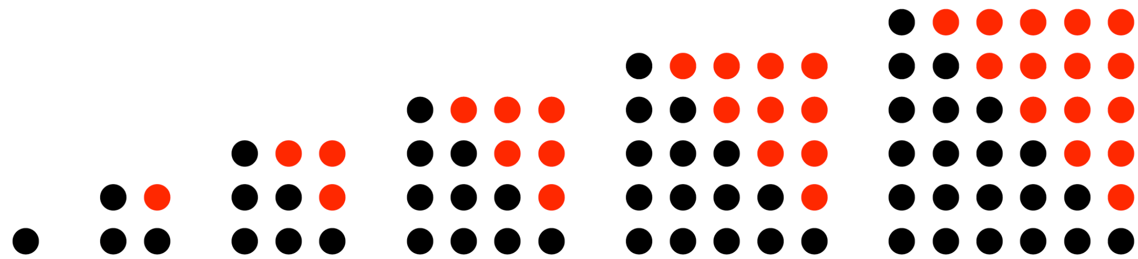
Always try to value and honor how the students see their patterns!

3. A right triangle arrangement of dots can also display the pattern of triangular numbers conveniently:



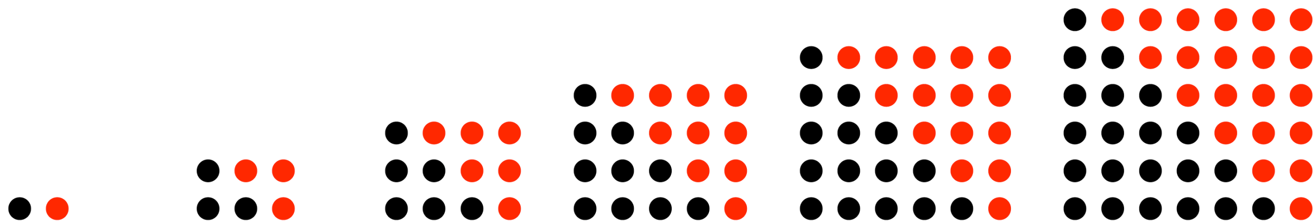
Does this particular display help you think of any additional patterns regarding triangular numbers?

This particular arrangement of Triangular Number arrays can prompt students to see that, if you put two adjacent triangular numbers together, the dots in the new array will always form a square.



T_1 $T_1 + T_2$ $T_2 + T_3$ $T_3 + T_4$ $T_4 + T_5$ $T_5 + T_6$

Another pattern students might see is that if they duplicate, rotate, and build a rectangle with the second array of dots, a special rectangle, called an oblong, is created. The dimensions of the oblong will be n x $(n + 1)$ or $n^2 + n$.



1x2 or 1x1+1 *2x3 or 2x2=2* *3x4 or 3x3+3* *4x5 or 4x4+4* *5x6 or 5x5+5* *6x7 or 6x6+6*

General Forms: **n x $(n + 1)$** **or** **$n^2 + n$**

4. Can you apply any of the techniques used in solving ***The Handshake Problem*** or ***All Possible Diagonals*** to this problem? What are some common elements among the three problems?

Each of these three problems contains the same additive pattern: as the first term increases by 1, the second term increases by the next counting number. For example, when there is only 1 person, there are zero handshakes. With 2 people, you have one handshake (0 + 1); 3 people create three handshakes (add 2 to 1, the previous number of handshakes). When 4 people gather, six handshakes happen (add 3 to 3). For the number of sides and diagonals and for the total number of dots in a triangular array, the same recursive pattern occurs: first add 1, then add 2, then add 3, and so on.

Each of the general forms for the nth term uses the multiplication of two expressions that involve n and division by two.

Assessment Options:

If students have not yet seen Gaussian summation, you might use the following as your next bonus problem since one of the classic solutions to the Gaussian sum involves "double counting" the sum and dividing by two to get the actual answer much as we can double count the triangular numbers in an oblong array (question #3 above).

> Sum the numbers, 1 + 2 + 3 + 4 + 5 + … + 9 + 10;
> 1 + 2 + 3 + 4 + 5 + … + 29 + 30; and/or
> 1 + 2 + 3 + 4 + 5 + … + 99 + 100,
> showing your thinking as well as your answer.
>
> You will receive *double* bonus points if you can generalize the summation of the numbers from 1 to *n*!

Students might use a method such as:

$$\begin{array}{l} 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\ \underline{10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1} \\ 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 + 11 \end{array}$$

= 11 x 10 or 110, which is exactly twice the sum we want. Therefore, we divide by two: 110 / 2 = **55**.

$$\begin{array}{l} 1 + 2 + 3 + \dots + 28 + 29 + 30 \\ \underline{30 + 29 + 28 + \dots + 3 + 2 + 1} \\ 31 + 31 + 31 + \dots + 31 + 31 + 31 \end{array}$$

= 31 x 30 or 930, which is exactly twice the sum we want. Therefore, we divide by two: 930 / 2 = **465**.

$$\begin{array}{l} 1 + 2 + 3 + \dots + 98 + 99 + 100 \\ \underline{100 + 99 + 98 + \dots + 3 + 2 + 1} \\ 101 + 101 + 101 + \dots + 101 + 101 + 101 \end{array}$$

= 101 x 100 or 10100, which is exactly twice the sum we want. Therefore, we divide by two: 10100 / 2 = **5050**.

$$\begin{array}{l} 1 + 2 + 3 + \dots + (n-2) + (n-1) + n \\ \underline{n + (n-1) + (n-2) + \dots + 3 + 2 + 1} \\ (n+1) + (n+1) + (n+1) + \dots + (n+1) + (n+1) + (n+1) \end{array}$$

= (n + 1) x n, which is exactly twice the sum we want. Therefore, we divide by two: ½ (n + 1) x n.

Alternately, you might use one of the **Extensions** for an assessment question.

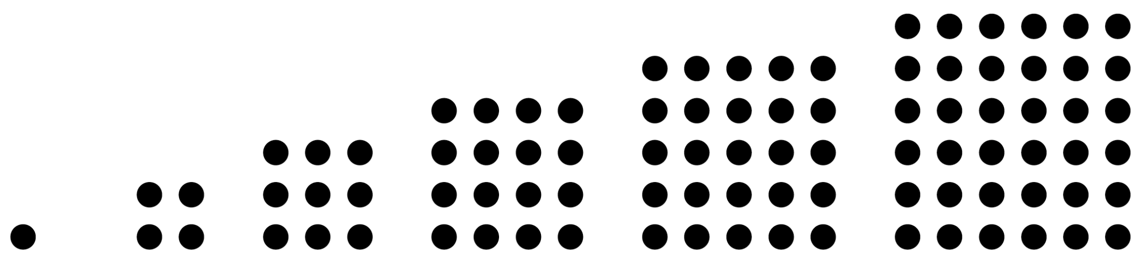
Extensions:

1. Find a general formula for the sum the first *n* odd numbers. (If you need a hint to get started, make a table and find the sums of the first several odd numbers and look for a pattern.)

n	Sum of the first n odd numbers
1	**1**
2	1 + 3 = **4**
3	1 + 3 + 5 = **9**
4	1 + 3 + 5 + 7 = **16**
5 …	1 + 3 + 5 + 7 + 9 = **25**
n	$\mathbf{n^2}$

2. Explore other figurative numbers that can be represented geometrically and whose *n*th term can be generalized. You might start with *Square Numbers*, which can be represented by dots in a square arrangement or array, and *Oblong Numbers*, which can be represented in a rectangular array having one dimension one unit longer than the other. Even more challenging would be *Pentagonal Numbers* and *Hexagonal Numbers*, which can be represented by dots in a pentagonal array or hexagonal array respectively.

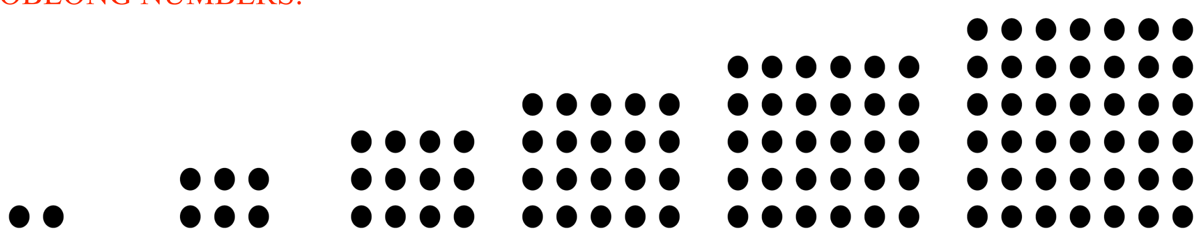
SQUARE NUMBERS:



n=1, T=1 *n=2, T=4* *n=3, T=9* *n=4, T=16* *n=5, T=25* *n=6, T=36*

In general, nth term is $\mathbf{n^2}$.

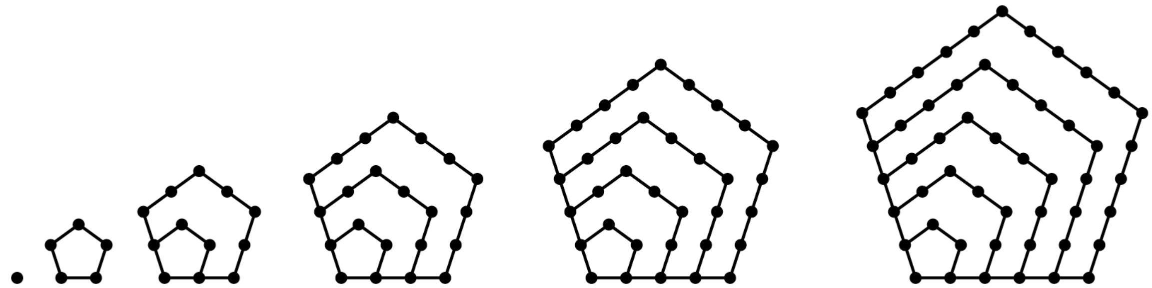
OBLONG NUMBERS:



n=1, T=2 *n=2, T=6* *n=3, T=12* *n=4, T=20* *n=5, T=30* *n=6, T=42*

In general, nth term is $\boldsymbol{n \times (n + 1)}$ or $\boldsymbol{n^2 + n}$.

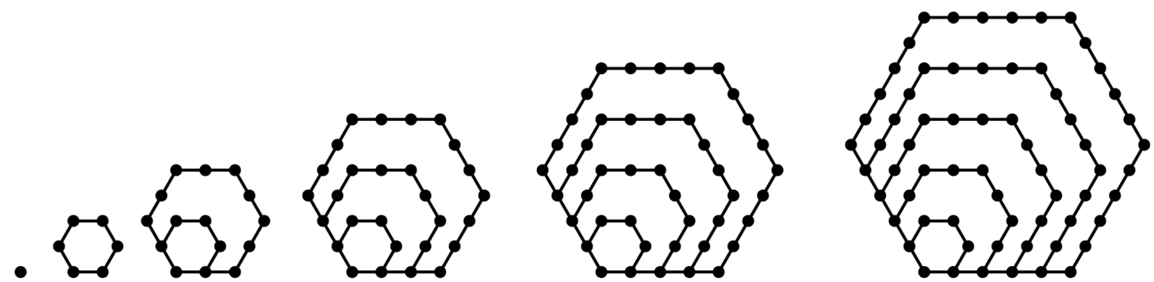
PENTAGONAL NUMBERS:



n = 1, *T* = 1 *n* = 2, *T* = 5 *n* = 3, *T* = 12 *n* = 4, *T* = 22 *n* = 5, *T* =35 *n* = 6, *T* = 51

In general, nth term is $\frac{1}{2} n \times (3n - 1)$.

HEXAGONAL NUMBERS:



n = 1, *T* = 1 *n* = 2, *T* = 6 *n* = 3, *T* = 15 *n* = 4, *T* = 28 *n* = 5, *T* = 45 *n* = 6, *T* = 66

In general, nth term is $n \times (2n - 1)$.

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