Distances: What Is Like a Square?

1. Draw four points in the plane, say A, B, C, and D, located at the corners of a square. Any two of the four points have a certain distance between them, so there are six distances: AB, AC, AD, BC, BD, and CD. But in the case of a square, many of those distances are equal. How many *distinct* distances are there?

2. Can you find another arrangement of four points in the plane, not a square, that has this same property? In other words, can you find a different set of four points that has the same number of distinct distances as a square?

3. How about yet another arrangement? How many can you find?

4. If you have five points in the plane, what is the fewest distinct distances they can have? How do you arrange them to obtain it?

5. a) How many points in the plane can be arranged so that there's only one distance between any pair of points?
   b) What is the answer if the points are allowed to be in 3D instead of the plane?

6. In the next several problems, we'll look at points arranged in one dimension, along a single line.
   Look at all the possible arrangements of three points along a line. What is the fewest number of distinct distances? What is the most?

7. Now move on to four points on a line. Show that if there are 3 distinct distances, there's only one configuration possible.

8. Still with four points on a line, take a look at the configurations with 4 distinct distances. Try to describe all possible configurations. Can you guarantee that you found all the possibilities?

9. Going back to the original question: four points on the plane, two distinct distances. Look for a systematic, organized way to find all the possible arrangements. Prove that your list is complete.

10. (Hard) Can you repeat that for five points in the plane with two distinct distances? With three?

11. (Really hard) Determine the "size" (or "dimension") of n-point arrangements in d-dimensional space with k distinct distances. As a first step, determine which values of n, k, and d are "illegal" in the sense that no arrangement is possible.