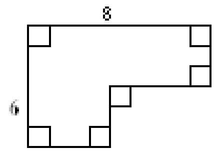
The Teacher's Circle, August 14 – 18, 2006

I. EASY PROBLEMS

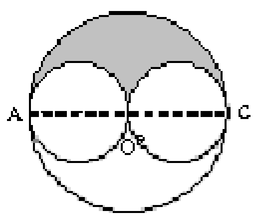
1. (AJHSME, 1986)



The perimeter of the polygon shown is

A) 14 B) 20 C) 28 D) 48
E) cannot be determined from the information given

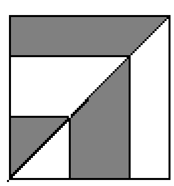
2. (AJHSME, 1986)



The larger circle has diameter AC. The two small circles have their centers on AC and just touch at O, the center of the large circle. If each small circle has radius 1, what is the value of the ratio of the area of the shaded region to the area of one of the small circles?

A) between _ and 1 B) 1 C) between 1 and 3/2 D) between 3/2 and 2
E) cannot be determined from the information given

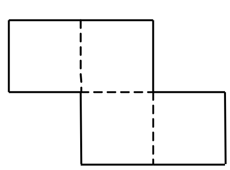
3. (AJHSME 1990)



What fraction of the square is shaded?

A) 1/3 B) 2/5 C) 5/12 D) 3/7 E) _

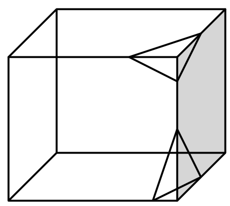
4. (AJHSME 1990)



The area of this figure is 100 cm^2. (The figure consists of four identical squares.) Its perimeter is

A) 20 cm B) 25 cm C) 30 cm D) 40 cm E) 50 cm

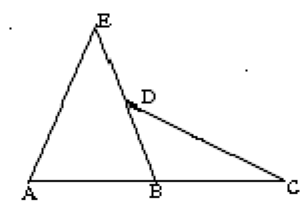
5. (AJHSME 1990)



Each corner of a rectangular prism is cut off. Two (of the eight) cuts are shown. How many edges does the new figure have?

A) 24 B) 30 C) 36 D) 42 E) 48

6. (AJHSME 1994)

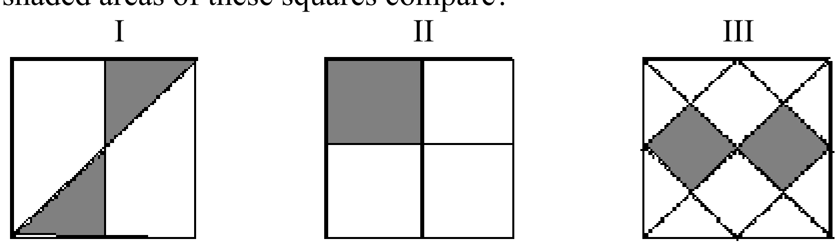


If $\angle A = 60°$, $\angle E = 40°$ and $\angle C = 30°$, then $\angle BDC =$

A) 40° B) 50° C) 60° D) 70° E) 80°

7. (AJHSME 1994)

Each of the three large squares shown below is the same size. Segments that intersect the sides of the squares intersect at the midpoints of the sides. How do the shaded areas of these squares compare?



A) The shaded areas in all three are equal.
B) Only the shaded areas of I and II are equal.
C) Only the shaded areas of I and III are equal.
D) Only the shaded areas of II and III are equal.

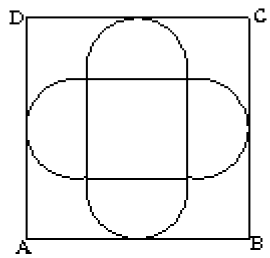
E) The shaded areas of I, II and III are all different.

8. (AJHSME 1994)

The perimeter of one square is 3 times the perimeter of another square. The area of the larger square is how many times the area of the smaller square?

A) 2 B) 3 C) 4 D) 6 E) 9

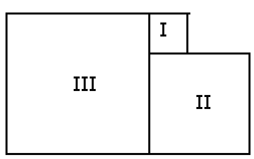
9. (AJHSME 1994)



Around the outside of a 4 by 4 square, construct four semicircles (as shown in the figure) with the four sides of the square as their diameters. Another square, ABCD, has its sides parallel to the corresponding sides of the original square, and each side of ABCD is tangent to one of the semicircles. The area of the square ABCD is

A) 16 B) 32 C) 36 D) 48 E) 64

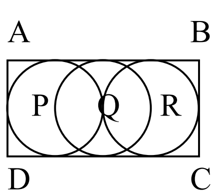
10. (AJHSME 1995)



Figures I, II and III are squares. The perimeter of I is 12 and the perimeter of II is 24. The perimeter of III is

A) 9 B) 18 C) 36 D) 72 E) 81

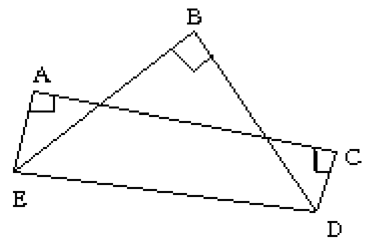
11. (AJHSME 1995)



Three congruent circles with centers P, Q and R are tangent to the sides of rectangle ABCD as shown. The circle centered at Q has diameter 4 and passes through points P and R. The area of the rectangle is

A) 16 B) 24 C) 32 D) 64 E) 128

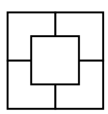
12. (AJHSME 1995)



In the figure, $\angle A$, $\angle B$, and $\angle C$ are right angles. If $\angle AEB = 40°$ and $\angle BED = \angle BDE$, then $\angle CDE =$

A) 75° B) 80° C) 85° D) 90° E) 95°

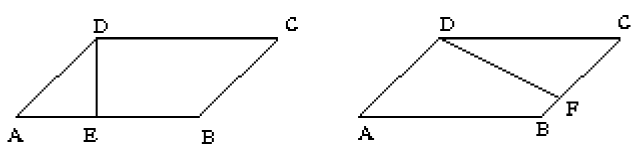
13. (AJHSME 1995)



The area of each of the four congruent L-shaped regions of this 100-inch by 100-inch square is 3/16 of the total area. How many inches long is the side of the center square?

A) 25 B) 44 C) 50 D) 62 E) 75

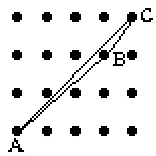
14. (AJHSME 1995)



In parallelogram ABCD, DE is the altitude to the base AB and DF is the altitude to the base BC. [**Note**: *Both pictures represent the same parallelogram*.] If DC =12, EB = 4 and DE = 6, then DF =

A) 6.4 B) 7 C) 7.2 D) 8 E) 10

15. (AJHSME 1996)



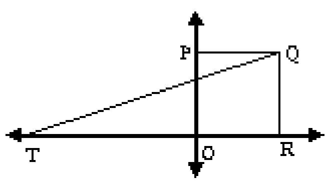
The horizontal and vertical distances between adjacent points equal 1 unit. The area of triangle ABC is

A) _ B) _ C) _ D) 1 E) 5/4

16. (AJHSME 1996)
Points A and B are 10 units apart. Points B and C are 4 units apart. Points C and D are 3 units apart. If A and D are as close as possible, then the number of units between them is

A) 0 B) 3 C) 9 D) 11 E) 17

17. (AJHSME 1996)

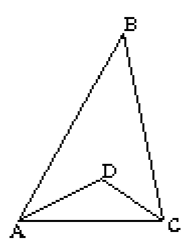


NOT TO SCALE

Figure OPQR is a square. Point O is the origin, and point Q has coordinates (2, 2). What are the coordinates for T so that the area of triangle PQT equals the area of square OPQR?

A) $(-6,0)$ B) $(-4,0)$ C) $(-2,0)$ D) $(2,0)$ E) $(4,0)$

18. (AJHSME 1996)



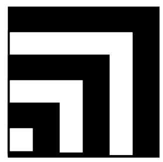
The measure of angle ABC is $50°$, AD bisects angle BAC, and DC bisects angle BCA. The measure of angle ADC is

A) 90° B) 100° C) 115° D) 122.5° E) 125°

19. (AJHSME 1997)
The area of the smallest square that will contain a circle of radius 4 is

A) 8 B) 16 C) 32 D) 64 E) 128

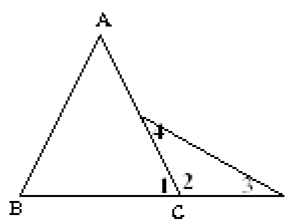
20. (AJHSME 1997)



What fraction of this square region is shaded? Stripes are equal in width, and the figure is drawn to scale.

A) 5/12 B) _ C) 7/12 D) 2/3 E) 5/6

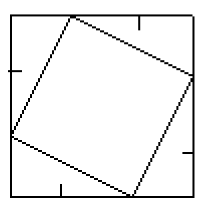
21. (AJHSME 1997)



$\angle ABC = 70°$, $\angle BAC = 40°$, $\angle 1 + \angle 2 = 180°$, $\angle 3 = \angle 4$. Find $\angle 4$

A) 20° B) 25° C) 30° D) 35° E) 40°

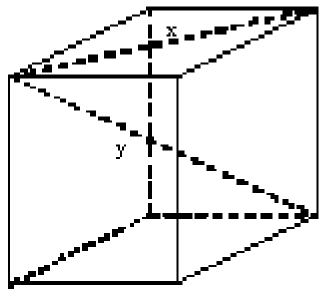
22. (AJHSME 1997)



Each side of the large square in the figure is trisected (divided into three equal parts). The corners of an inscribed square are at these trisection points, as shown. The ratio of the area of the inscribed square to the area of the large square is

A) $\sqrt{3}/3$ B) 5/9 C) 2/3 D) $\sqrt{5}/3$ E) 7/9

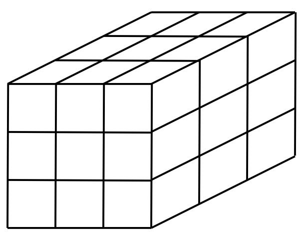
23. (AJHSME 1997)



A cube has eight vertices (corners) and twelve edges. A segment, such as *x*, which joins two vertices not joined by an edge is called a diagonal. Segment *y* is also a diagonal. How many diagonals does a cube have?

A) 6 B) 8 C) 12 D) 14 E) 16

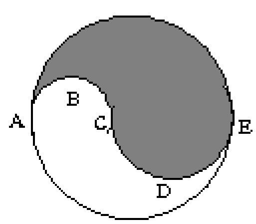
24. (AJHSME 1997)



Each corner cube is removed from this 3 cm x 3 cm x 3 cm cube. The surface area of the remaining figure is

A) 19 sq. cm B) 24 sq. cm C) 30 sq. cm D) 54 sq. cm E) 72 sq. cm

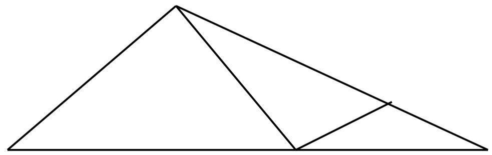
25. (AJHSME 1997)



Diameter ACE is divided at C in the ratio 2:3. The two semicircles, ABC and CDE, divide the circular region into an upper (shaded) region and a lower region. The ratio of the area of the upper region to that of the lower region is

A) 2:3 B) 1:1 C) 3:2 D) 9:4 E) 5:2

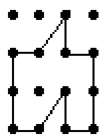
26. (AJHSME 1998)



How many triangles are in this figure? (Some triangles may overlap other triangles.)

A) 9 B) 8 C) 7 D) 6 E) 5

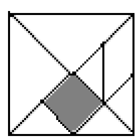
27. (AJHSME 1998)



Dots are spaced one unit apart, horizontally and vertically. The number of square units enclosed by the polygon is

A) 5 B) 6 C) 7 D) 8 E) 9

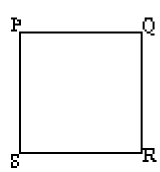
28. (AJHSME 1998)



What is the ratio of the area of the shaded square to the area of the large square? (The figure is drawn to scale.)

A) 1/6 B) 1/7 C) 1/8 D) 1/12 E) 1/16

29. (AJHSME 1998)



Let PQRS be a square piece of paper. P is folded onto R and then Q is folded onto S. The area of the resulting figure is 9 square inches. Find the perimeter of square PQRS.

A) 9 B) 16 C) 18 D) 24 E) 36

30. (AJHSME 1998)

A 4 x 4 x 4 cubical box contains 64 identical small cubes that exactly fill the box. How many of these small cubes touch a side or the bottom of the box?

A) 48 B) 52 C) 60 D) 64 E) 80

31. (AMC 8 1999)

What is the degree measure of the smaller angle formed by the hands of a clock at 10 o'clock?

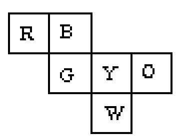
A) 30 B) 45 C) 60 D) 75 E) 90

32. (AMC 8 1999)

A rectangular garden 50 feet long and 10 feet wide is enclosed by a fence. To make the garden larger, while using the same fence, its shape is changed to a square. By how many square feet does this enlarge the garden?

A) 100 B) 200 C) 300 D) 400 E) 500

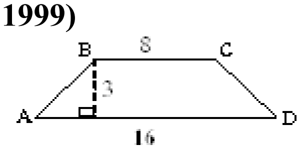
33. (AMC 8 1999)



Six squares are colored, front and back, (R=red, B=blue, O=orange, Y=yellow, G=green, and W=white). They are hinged together as shown, then folded to form a cube. The face opposite the white face is

A) B B) G C) O D) R E) Y

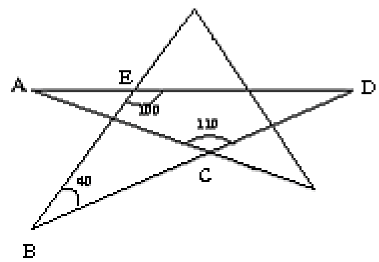
34. (AMC 8 1999)



In trapezoid ABCD, the sides AB and CD are equal. The perimeter of ABCD is

A) 27 B) 30 C) 32 D) 34 E) 48

35. (AMC 8 1999)

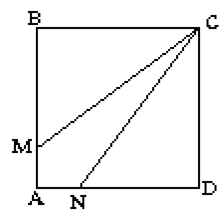


$\angle B = 40°$, $\angle BED = 100°$, $\angle ACD = 110°$. The degree measure of angle A is

A) 20 B) 30 C) 35 D) 40 E) 45

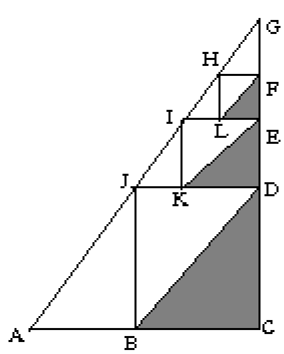
36. (AMC 8 1999)

Square ABCD has sides of length 3. Segments CM and CN divide the square's area into three equal parts. How long is segment CM?



A) $\sqrt{10}$ B) $\sqrt{12}$ C) $\sqrt{13}$ D) $\sqrt{14}$ E) $\sqrt{15}$

37. (AMC 8 1999)



Points B, D, and J are midpoints of the sides of right triangle ACG. Points K, E, I are midpoints of the sides of triangle JDG, etc. If the dividing and shading process is done 100 times (the first three are shown) and AC = CG = 6, then the total area of the shaded triangles is nearest

A) 6 B) 7 C) 8 D) 9 E) 10

38. (AMC 8 2000)

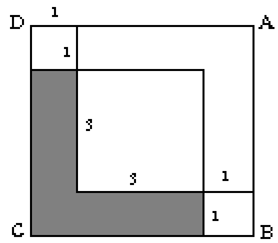
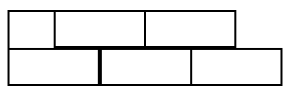


Figure ABCD is a square. Inside this square three smaller squares are drawn with side lengths as labeled. The area of the shaded L-shaped region is

A) 7   B) 10   C) 12.5   D) 14   E) 15

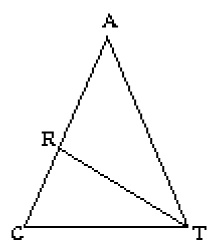
39.   (AMC 8 2000)

A block wall 100 feet long and 7 feet high will be constructed using blocks that are 1 foot high and either 2 feet long or 1 foot long (no blocks may be cut). The vertical joins in the blocks must be staggered as shown, and the wall must be even on the ends. What is the smallest number of blocks needed to build this wall?



A) 344   B) 347   C) 350   D) 353   E) 356

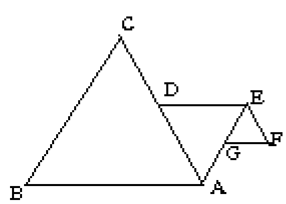
40.   (AMC 8 2000)



In triangle CAT, we have $\angle ACT = \angle ATC$ and $\angle CAT = 36°$. If TR bisects $\angle ATC$, then $\angle CRT =$

A) 36°   B) 54°   C) 72°   D) 90°   E) 108°

41.   (AMC 8 2000)



Triangles ABC, ADE, and EFG are all equilateral. Points D and G are midpoints of AC and AE, respectively. If AB = 4, what is the perimeter of figure ABCDEFG?

A) 12 B) 13 C) 15 D) 18 E) 21

42. (AMC 8 2000)

In order for Mateen to walk a kilometer (1000 m) in his rectangular backyard, he must walk the length 25 times or walk its perimeter 10 times. What is the area of Mateen's backyard in square meters?

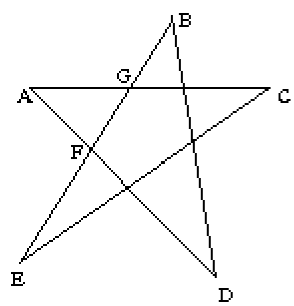
A) 40 B) 200 C) 400 D) 500 E) 1000

43. (AMC 8 2000)

A cube has edge length 2. Suppose that we glue a cube of edge length 1 on top of the big cube so that one of its faces rests entirely on the top face of the larger cube. The percent increase in the surface area (sides, top, and bottom) from the original cube to the new solid formed is closest to:

A) 10 B) 15 C) 17 D) 21 E) 25

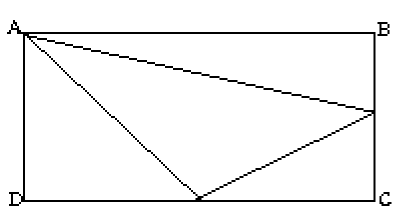
44. (AMC 8 2000)



If $\angle A = 20°$ and $\angle AFG = \angle AGF$, then $\angle B + \angle D =$

A) $48°$ B) $60°$ C) $72°$ D) $80°$ E) $90°$

45. (AMC 8 2000)

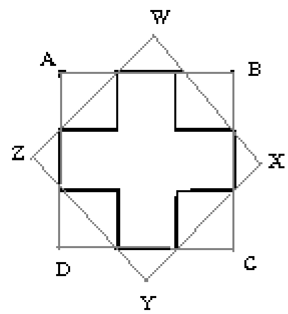


The area of rectangle ABCD is 72. If point A and the midpoints of BC and CD are joined to form a triangle, the area of that triangle is

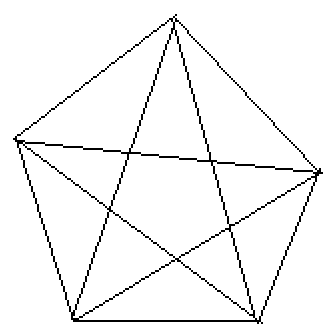
A) 21 B) 27 C) 30 D) 36 E) 40

II. INTERMEDIATE PROBLEMS

1. A cross shape with twelve edges, each of length 1, is inscribed in two squares ABCD and WXYZ, as shown. Find the area of each of these two squares.



2. The diagram shows a **regular** pentagon, a pentagram (the five-pointed star) and a smaller pentagon inside. The perimeter of the inner pentagon is 5 cm and the perimeter of the pentagram is $10x$ cm. Show that the perimeter of the outer pentagon is $5x^2$ cm.



NOT TO SCALE

3. The rectangle ABCD has AB = 15 and AD = 10. P is the point inside the rectangle for which AP =10 and DP = 12. Find the angle DPC.

4. Triangle ABC has AB = AC = 5 cm and BC = 6 cm. From any point P, inside or on the boundary of this triangle, line segments are drawn at right angles to the sides; the lengths of these line segments are x, y and z cm.

(a) Find the largest possible value of the total $x + y + z$ and find the positions of P where this largest total occurs.

(b) Find the smallest value of the total $x + y + z$ and find the positions of P where this smallest total occurs.

(c) What if ABC is a general (non-isosceles) triangle?

5. In triangle ABC, $\angle ACB$ is a right angle, BC = 12 and D is a point on AC such that AD = 7 and DC = 9. The perpendicular from D to AB meets AB at P and the perpendicular from C to BD meets BD at Q. Calculate:

(a) The ratio of the area of triangle BCD to the area of triangle BAD.

(b) The ratio of the length of QC to the length of PD.

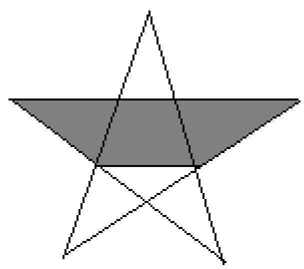
6. A square has one corner folded over to create a pentagon. The three shorter sides of the pentagon which is formed are all the same length. Find the area of the pentagon as a fraction of the area of the original square.

7. A square is inscribed inside a quadrant of a circle of radius 10 cm. Calculate the area of the square.

8. In a triangle the length of one side is 3.8 cm and the length of another side is 0.6 cm. Find the length of the third side if it is known that it is an integer (when expressed in centimeters).

9. Using a pen and a straight edge, draw on a square grid a rectangle whose area is
(a) twice the area of one square of the grid;
(b) 5 times the area of one square of the grid.

10. Find the area of the shaded region as a fraction of the area of the entire regular pentagram.



NOT TO SCALE

11. Is it true that a half liter bottle of Coke is proportional to a liter bottle of Coke, i.e., one can be obtained from another by multiplying all lengths by the same factor?

III. HARDER PROBLEMS

1. In a certain country, there are 100 airports and all the distances between them are different. An airplane takes off from each airport and lands at the closest airport. Prove that none of the 100 airports receives more than 5 planes.

2. n points are placed in a plane in such a way that the area of every triangle with vertices at any three of these points is at most 1. Prove that all these points can be covered by a triangle with the area of 4.

3. Is it possible to place 1000 line segments in a plane in such a way that every endpoint of each of these line segments is at the same time an inner point of another segment?

4. Suppose that points A, B, C, and D are coplanar but non-collinear. Prove that at least one of the triangles formed by these points is not acute.

5. In a coordinate plane there are infinitely many rectangles. Vertices of every rectangle have coordinates (0, 0), (0, m), (n, 0), (n, m) where m and n are positive integers (different for different rectangles). Prove that it is possible to choose two of these rectangles so that one is completely covered by the other.

We will need to recall the meaning of the term 'convex'. A *convex* planar figure is the intersection of a number (finite or infinite) of half-planes. The intersection of a finite number of half-planes is a *convex polygon*. (Equivalently, a figure is convex if for any two points A and B of the figure, the entire line segment AB belongs to the figure.)

6. Prove that

(a) any convex polygon of area 1 can be covered by a parallelogram of area 2;

(b) a triangle of area 1 cannot be covered by a parallelogram of area less than 2.

7. (a) Suppose that there are four convex figures in a plane, and every three of them have a common point. Prove that all four figures have a common point.

(b) Suppose that there are n convex figures in a plane, and every three of them have a common point. Is it necessarily true that all n figures have a common point?

8. A number of lines segments lie in a plane in such a way that for any three of them there exists a line intersecting them. Prove that there exists a line intersecting all these segments.

9. Is it true that for every pentagon it is possible to find at least two sides such that the pentagon belongs to the intersection of exactly two half-planes determined by these sides.

10. (a) Draw a polygon and a point P inside this polygon so that none of the sides is completely visible from the point P.

(b) Draw a polygon and a point P outside of this polygon so that none of the sides is completely visible from the point P.

11. (a) Prove that every n-gon (with $n \geq 4$) has at least one diagonal that is completely contained inside the n-gon.

(b) Find the least possible number of such diagonals.

(c) Prove that every polygon can be cut into triangles by non-intersecting diagonals.

(d) Suppose that a polygon is cut into triangles by non-intersecting diagonals. Prove that it is possible to color the vertices of the polygon using three colors in such a way that all three vertices of each of the triangles are different?

12. Is it possible to cover a 10-by-10 'chessboard' by 25 1-by-4 dominos?