Grid Power

Tatiana Shubin tatiana.shubin@sjsu.edu

Warm-up problems:

- Is it possible to place dots into squares of an 8×8 grid (no more than one dot per square) so that the number of dots in every column is the same, while no two rows have the same number of dots?

- Is it possible to cover all squares of a chessboard with 32 dominos (each domino covering exactly two squares) in such a way that no two dominos cover a 2 by 2 square?

1. (a) How many squares are there in a 7×7 square? 5×8 rectangle? $m \times n$ rectangle?
   (b) How many rectangles are there in a 7×7 square? 5×8 rectangle? $m \times n$ rectangle?

2. On your grid paper, draw a rectangle and one of its diagonals. How many grid squares are crossed by the diagonal?

3. Draw a rectangle and trace the path of a billiard ball that begins in the lower left corner and initially travels upward at a 45-degree angle. Assuming that the ball bounces off the walls at perfect 45-degree angles, which corner does the ball reach first? What fraction of all the unit squares within your rectangle does the ball pass through on its way? Start your experiment with a rectangle having width 3 and height 5, then choose other dimensions. What will happen if the width is m and the height n units for some positive integers m and n?

4. Draw a square and connect each vertex with the midpoints of two opposite sides (8 lines altogether). How many right triangles can you find in this diagram? How many of these triangles are 3-4-5 triangles? Can you prove that they indeed are of this type?

5. (a) Is it possible to split a square into 2 squares? 4 squares? 6 squares? 7 squares? Can you decide for which positive integers n it is possible to split a given square into n smaller squares? (Smaller squares don't need to be of the same size.)

(b) Is it possible to split a rectangle (or even a square) into a number of non-congruent squares?

For the next three problems try to imagine painting each of the grid intersection (a.k.a. *lattice point*) of an infinite sheet of grid paper by one of a given number of colors.

6. If each integer of a number line is painted with one of two colors, is it possible to do it so that there are no three *monochromatic* (this means 'of the same color') numbers forming an arithmetic progression?

7. If each lattice point of a plane is painted by one of two colors, is it possible to avoid getting three monochromatic points which form an isosceles right triangle? What if you can use three different colors instead of two?

8. Suppose that, again, you paint each lattice point with one of two colors. Can you do this in such a way that you avoid producing four monochromatic points forming a square?

The last two items are games:

9. On one square of an 8 by 8 chessboard there is a "lame rook" that can move either to the left or down through any number of squares. Two players take turns moving the rook. The player unable to move the rook loses. Which of the two players has a winning strategy? (Consider various initial positions of the rook.)

10. And don't forget the Infinite Tic-Tac-Toe!