Foundations of tropical schemes

organized by
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Workshop Summary

Introduction. Tropical geometry is an emerging area at the interface between algebraic geometry and combinatorics, in which an algebraic variety is replaced by a tropical variety, which is a polyhedral object. The tropical variety can be viewed as a "combinatorial shadow" and used to recover various numerical and algebraic invariants of the original geometric object. For the first decade of its development, the subject concentrated on applications to varieties and cycles. However, a number of problems in algebraic geometry, due to their scheme-theoretic nature, have remained beyond the scope of the various combinatorial tools developed. Tropical scheme theory seeks to generalize the standard constructions in scheme theory to semirings, with the aim of providing algebro-geometric foundations for the combinatorial notions appearing in tropical geometry. The first steps in this program appear in work of Giansiracusa-Giansiracusa, where a method of tropicalizing a scheme is introduced, of Maclagan-Rincón, where the connection to combinatorics and valuated matroids is elaborated, and of Macpherson, where connections to non-archimedean geometry are emphasized. Further exciting developments in the area by several other groups followed quickly.

These ideas and their interactions with related fields were explored during a Spring 2014 workshop at Yale organized by Sam Payne. This AIM Workshop served as a natural continuation—and expansion—of that Yale meeting, bringing several groups together to pursue the development of tropical scheme theory and encourage interaction between related fields and researchers with a view towards fruitful applications to outstanding problems in classical algebraic geometry. We were extremely pleased with the opportunity provided by AIM to bring these people together, particularly for a workshop creating active working groups to study specific problems of interest, complemented by lively discussions on long-term goals and directions to develop this burgeoning field.

Talks. In standard AIM fashion, the mornings were reserved for talks. We emphasized broad survey talks aimed at bringing the entire group up to speed on the topics of the workshop.

- The week began on Monday with talks by Jeffrey Giansiracusa and Felipe Rincón. Giansiracusa gave a broad introduction to the general area of tropical schemes, focusing on the constructions of Giansiracusa-Giansiracusa and their algebro-geometric approach to studying bend relations in polynomial algebras over the tropical semiring. Rincón continued, focusing on the relationship between matroids and the commutative algebra in this theory. Rincón's talk ended with a series of fundamental open questions about tropical ideals and tropical commutative algebra.
- On Tuesday, Oliver Lorcheid began with a broad overview of the theory of Blueprints and Blue Schemes, concluding with his result that schemetheoretic tropicalization

can be viewed as a base-change in this setting. This was followed by a survey on logarithmic geometry and logarithmic methods for tropicalization by Martin Ulirsch.

- Wednesday's talks by Kalina Mincheva and Dhruv Ranganathan featured a mix of foundational topics and applications. Mincheva began the morning providing an introduction to the study of prime congruences and the dimension theory they produce in polynomial algebras over the tropical semiring (based on her joint work with Dániel Joó).

  Ranganathan continued the morning providing applications of the theory of Artin fans coming from logarithmic geometry to realizability problems for tropical stable maps. Ranganathan ended his talk providing a possibly related application for structures from tropical scheme theory to realizability problems for tropical linear series.
- Thursday morning began with a survey talk by Josephine Yu on tropical linear spaces and valuated matroids. This was followed by a talk by Jonathan Wise on tropical stacks with a particular view towards the tropicalization of the moduli space of curves. Wise's talk was based on a preprint that had been posted to the arXiv during the week, and that was the first output of an AIM SQUARE group.

Monday afternoon began with a problem session, with an over-arching goal of guiding the development of tropical scheme theory for the next five to ten years. For the first part of the session we focused in particular on questions in algebraic geometry that do not ostensibly concern tropical geometry but for which tropical scheme-theoretic techniques may eventually apply. The afternoon finished with three brief tutorial sessions (one on matroids, one on Hilbert schemes of points, and one on basic examples/exercises in tropical scheme theory) to prepare participants for working groups throughout the rest of the week.

Working groups. In the afternoons on Tuesday through Thursday, and also Friday morning, we broke into working groups following the AIM protocol for a two-round voting process. Many groups reformed and changed scope during the week so it is difficult to count them precisely, but we estimate the figure at a total of 14 groups on the following topics:

(1) *What is a good notion of a prime tropical ideal?* One outcome of this group was the observation that a tropical ideal is prime in the "naive" algebraic sense only if it is the ideal of a single point.
(2) *Understand the tropicalization of the Hilbert scheme from a scheme-theoretic perspective.* This group focused particularly on the tropicalization of the Hilbert scheme of points.
(3) *Merge the theories of tropical schemes and abstract tropical curves.* One approach this group took was to use the universal tropicalization introduced by Giansiracusa-Giansiracusa as a scheme-theoretic replacement for the Berkovich analytification and then look for skeletons as closed subschemes; it was determined that methods of Hrushovski-Loeser may assist with this.
(4) *How can we describe a tropical ideal with a finite amount of information?* This led to one of the more unexpected outcomes of the workshop, as this group showed that this was not possible, at least in a naive sense, but that the failure of finiteness leads to the infinitely branching tree structures seen in non-archimedean analytic geometry.
(5) *Tropicalize subschemes of abelian varieties.* This was motivated by the problem of finding a tropical scheme structure on subvarieties of the Jacobian of a tropical curve.

As currently understood, scheme-theoretic tropicalization is defined for subvarieties of toric varieties. Extending this to abelian varieties could potentially have far-reaching applications and requires novel insight into the topology of semiring schemes.

(6) *What is the definition of a flat family of tropical schemes?*

Grappling with this question led this group to the study of algebraic families of matroids and in particular to the task of introducing a valuated analogue of the lattice of flats cryptomorphism for matroids.

(7) *Develop Čech cohomology for tropical schemes.* This group explored a definition of the Čech differential based on tropical kernels and bend relations. They showed that the image is contained in the tropical kernel for this setting and developed a list of examples of sheaf cohomology computations to test this theory.

(8) *Find a tropical ideal with a non-realizable tropical variety.*

This group focused on finding a tropical ideal whose variety is the Bergman fan of a non-realizable matroid, such as the Vamos matroid.

(9) *What is the relationship between divisors and line bundles on tropical schemes?* This group found that the set-theoretic situation behaves as expected but that when enriching to scheme-structure one faces topological challenges that need to be address first. In order to make this exploration more tractable, the group sprouted off the following group:

(10) *Compute the Picard group of a plane elliptic curve, viewed as a tropical scheme.* Only begun on the last day, this group focused mostly on outlining the steps to take after AIM, such as producing a line bundle from each point of the curve and seeing what notion of local triviality is appropriate.

(11) *Compare log and blue schemes with a view towards moduli spaces.*

One outcome of this group was the insight that the intersection between log and blue schemes are those log schemes for which the log structure comes from maps to toric varieties.

(12) *How can we define the sum and intersection of tropical ideals?*

One of the workshop's goals was to develop basic commutative algebra in the tropical setting.

This foundational question would be the first step toward a useful theory.

(13) *When does a map from a curve to a tropical linear space determine a divisor?*

The relationship between divisors on curves and maps to projective space is very clear in the classical setting, but more mysterious in the tropical setting. This group determined a criterion for a tropical curve to map to a tropical linear space, showing that it is related to ideas such as tropical dependence, valuated matroids, and well-spacedness that have been studied earlier.

(14) *Consider lines on a cubic surface from a tropical scheme theory perspective.*

One of the early frustrations of tropical geometry is that there are smooth tropical cubic surfaces containing infinitely many lines, in contrast to the celebrated result from classical algebraic geometry that any such cubic surface contains exactly 27 lines. Some of the extraneous lines can be ruled out using tropical scheme structures, and this group considered the remaining family of mysterious lines, and whether log techniques could help here.

Presentations. On Friday afternoon these groups all gave short (5 minutes, with 3 minutes for questions) reports on their progress during the week, with a focus on presenting the roadblocks to full resolutions. It was both surprising and encouraging how much overlap there was between the groups on these issues; one outcome of the week is a clearer picture of the implications that will follow from, for example, a better understanding of tangent spaces in this context. During the reporting period Alex Fink also gave a longer report on the outcome of the "finite determination" group (group 4 above). This was an example of the overlap; this group showed that there was, in a naive sense at least, no finite amount of data that determines a tropical ideal by explicitly describing the tropicalization of the Hilbert scheme of two points in $\mathbb{P}^1$, which was also considered by the Hilbert schemes group.

Conclusion The workshop concluded with a brief problem session to explicitly list the "small technical" problems that we had realized during the week had broader implications. In addition, twelve potential collaboration groups formed during that afternoon and were listed on the white board to encourage continued efforts after AIM. The main outcomes of the workshop were

(1) Different mathematical communities that have been developing various forms of enriched tropical structures (blue schemes, Artin fans and log structures, tropical schemes) now have a common language and insight into each other's perspectives and programs, as well as awareness of common challenges and how cross-fertilization of ideas may help;
(2) Several specific groups (mixing members from the various sub-communities mentioned in the previous item, and also involving younger mathematicians eager to join a lively research community) formed during the week and solidified into focused collaborations that we expect to drive the development of tropical scheme theory in the years to come;
(3) The "finiteness" group's progress changed our perspective on how to approach tropical schemes from a computational perspective while also opening the door to a mysterious and potentially rich connection to Berkovich analytification;
(4) The problem sessions, both the lengthy opening one on Monday and the focused, technical one on Friday, provided motivation to continue intensively working on tropical scheme theory so that the foundational problems being explored now may be parlayed down the road be into tools to apply to outstanding problems in algebraic geometry.

We are grateful to AIM for hosting this workshop. This was a critical time in the development of this subfield to bring together a broad group of people, and we are optimistic that this investment will pay dividends in the near and long term.