Transversality in contact homology

organized by
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Workshop Summary

1. Overview

The main goal of this workshop was to evaluate, clarify, and extend the reach of traditional methods for establishing transversality in contact homology, allowing the wider community of researchers working on contact homologies in Europe, North America, South America, and Asia to come together to share their progress and discuss the foundational work needed for future developments. Talks and discussions were focused on the extent to which concrete geometric perturbations such as automatic transversality, obstruction bundle gluing, domain dependent almost complex structures can be used to obtain foundations for cylindrical, linearized, local and Legendrian contact homology. Also discussed was the relationship between these theories and symplectic homology.

The workshop made great progress identifying the current state of the field and clarifying the gaps in current arguments concerning the definition of contact homology. In particular Chris Wendl wrote a series of four detailed blog posts explaining the proofs of two main folk theorems [Wen1] - [Wen4], Michael Hutchings wrote a blog post clarifying the inability to define cylindrical contact homology with integer coefficients [Hut1], and Victor Ginzburg wrote a short article which was posted to the arXiv entitled "My Shopping List for Contact Homology" [Ginz1]. Additionally, many participants wrote extensive statements of interest with several different ideas to work on. Many of these were discussed at the workshop and have spawned new collaborations.

The organizers received much positive feedback from the participants and hope to have a "follow-up" workshop sometime in 2016. Many attendees expressed that the workshop was very productive and that they now have a much better understanding of the current state of various research projects. Overall we feel that this workshop was a definitive success.

2. Lectures

The morning lectures provided introductions to various topics of relevance. Hutchings gave an introduction to the status of the foundations of cylindrical contact homology, discussing when the classical definition is valid using generic almost complex structures. Bourgeois discussed his recent work with Oancea on the how to formulate S^1-equivariant symplectic homology for exact symplectic manifolds and its relationship with linearized contact homology when the latter can be defined [BO].

Nelson discussed her current work with Hutchings on obtaining invariance for cylindrical contact homology in dimension 3 for dynamically convex contact forms, and on constructing local contact homology using S^1-dependent almost complex structures and Morse-Bott methods [HN].

Abouzaid explained the use of piecewise geodesics to obtain a finite dimensional approximation of the free loop space. He then explained how these are used in the proof of Viterbo's theorem on cotangent bundles. These methods appeared in a similar guise during Ekholm's talk on his work with Bourgeois and Eliashberg on surgery formulas for Legendrian contact homology [BEE]. Ekholm also discussed special situations where one can make use of domain dependent complex structures to obtain sufficient transversality to define the terms in and prove the surgery formulas.

Wendl discussed the use of automatic transversality in dimension 4 which many participants found helpful as it is a very useful trick to obtain regularity for multiply covered curves [Wen0].

Hutchings introduced the technique of obstruction bundle gluing [HT] by using it to show that for dynamically convex contact forms in three dimensions, the non equivariant contact homology differential, defined using suitable correction terms, squares to 0.

Van Koert spoke about how S^1-equivariant symplectic homology can be used as a replacement for cylindrical contact homology to recover many applications of cylindrical contact homology in cases where its foundations are still lacking.

Hryniewicz and Ginzburg gave talks on local contact homology and pointed out the limitations of classical methods in establishing its foundations [HM].

3. Discussions

On the first afternoon there was an extensive Q&A session amongst newcomers and experts to clarify the issues in the construction of contact homology, which has been very confusing for those not working exclusively on that area but who want to use techniques and results. Each afternoon the organizers made sure a couple of experts were on hand to check in with the newcomers to field any questions they had from the morning lectures.

We had a large group discussion on the use of bifurcation analysis to obtain invariance of various kinds of contact homology, which began with Eliashberg posing some questions and then transformed into a large active discussion with Hutchings, Hryniewicz, Bourgeois, Ekholm, and van Koert all going up to the board to share their findings. The tentative conclusion is that proving invariance using contact homology is very difficult in general, but may be feasible in three dimensions.

Smaller group discussions continued throughout the week on the use of symplectic homology, what problems in local cylindrical contact homology could be resolved by current techniques, Legendrian surgery formulas, and the application of contact homology to embedding problems. Additionally,

- Bao discussed his work with Honda with various people during the week. They used obstruction bundle gluing to prove the chain homotopy equation in dimension 3 after one reduces the problem so that only hyperbolic noncontractible Reeb orbits remain in the chain groups.

- Morse theory in the presence of group actions was discussed throughout the week with the goal of developing a direct moduli space counting approach for extracting the invariant part of a homology theory, bypassing the construction of a group action on the complex. This group first developed a direct setup in finite dimensional toy models and tested it on cases in which equivariant Morse-Smale data exists. Then ways of applying abstract regularization techniques (e.g. polyfold theory) for

extending the approach to nontransverse situations were discussed - giving many participants first insights into ways of applying such abstract techniques. Eventually, we developed several outlines - both geometric and using abstract regularization - for constructing local contact homology as such invariant part of homology.

- Fish, Ekholm, and Bourgeois had a group discussion with many other participants on the use of Morse-Bott ideas in the study of pseudoholomorphic cylinders in symplectizations.

- Eliashberg shared his dreams on obtaining more general contact invariants.

- Bourgeois and Ekholm led a group discussion on the use of the graph trick to turn cylindrical contact homology into Legendrian contact homology. They also discussed conjectural relationships to questions raised by Shandon.

4. Specific Outcomes

The informal atmosphere of the workshop encouraged a great deal of discussion among the participants. Individuals were able to start new collaborations and discuss their ongoing collaborations during many of the small sessions.

Fish, Ginzburg, and Wehrheim started a project of demonstrating and testing the applicability of polyfold techniques to 'geometric dream proofs' at the example of a 'shopping list' construction by Ginzburg of local contact homology and its specific algebraic properties required for the detection of periodic orbits.

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