Relating test ideals and multiplier ideals

organized by
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Workshop Summary

To this day, the interplay of geometric methods in characteristic zero and Frobenius techniques in positive characteristic continues to inspire new questions and results throughout numerous areas of mathematics, including algebraic geometry, commutative algebra, representation theory, and number theory. This interaction arises via standard "reduction to characteristic p" procedures and yields fascinating connections. Nevertheless, our understanding of the implications of this process remains far from complete. The central focus of our workshop was to explore important open questions concerning the connection between two prominent and distinct means of measuring singularities which can be related in this manner: the *multiplier ideal* in complex algebraic geometry, and the *test ideal* in positive characteristic commutative algebra.

The participants included experts in commutative algebra, algebraic geometry, and arithmetic geometry. Many participants – particularly those working in different fields – were not well-acquainted with one another prior to the workshop, and several were in fact meeting in person for the first time. A broader goal of the workshop was to foster interaction and collaboration among researchers in these related but very distinct fields.

The first day of the workshop began with two survey talks. In the first talk, Manuel Blickle gave an overview of test ideals from a contemporary (rather than historical) perspective. In particular, his talk incorporated a perspective heavily motivated by duality and the Cartier operator. Next, Bhargav Bhatt gave an introduction to the connections between the main reduction conjecture for test ideals and various ordinarity conjectures in arithmetic geometry (notably introducing recent work of M. Mustaţă and V. Srinivas). In the afternoon, we began with a group discussion led by the organizers of the workshop. With ample input from the participants at large, we roughly summarized the current state of research concerning various workshop topics and presented a number of focus topics for the upcoming week. In particular, a number of open conjectures were highlighted throughout the discussion, as well clarification on various definitions. Roughly speaking, the topics broke down into three categories: non-$\mathbb{Q}$-Gorenstein questions, arithmetic questions, and questions concerning related numerical invariants (hand-written notes by Emily Witt can be found on the AIM website). The afternoon ended with a series of so-called "ask the expert" sessions: non-$\mathbb{Q}$-Gorenstein multiplier ideals by Tommaso de Fernex, definitions and examples by Kevin Tucker, and arithmetic connections by Bhargav Bhatt.

The second day of the workshop began much as the first, with two further survey talks. In the first talk, Anurag Singh gave a more historical overview of test ideals and tight closure from the perspective of commutative algebra. His talk also incorporated numerous examples, and filled in a number of gaps in the talks from the first day. Next, Mircea Mustaţă spoke on the known correspondence between test ideals and multiplier ideals via reduction

to characteristic p. In particular, he presented a simple proof in the case of an ideal of a regular ring starting from certain vanishing conditions for cohomology due to N. Hara and also to V. Mehta and V. Srinivas. In the afternoon, we first gathered a number of potential questions for working groups to consider. After voting, we broke off into groups to work on these questions. One group worked on connecting the multiplier ideal and test ideal in the non-$\mathbb{Q}$-Gorenstein setting, and seems to have made some partial progress right away (using a setting suggested in Anurag's talk from the morning). Other groups worked on trying to find congruence conditions for when the multiplier ideal reduces to the test ideal in simple settings, trying to understand Hartshorne-Speiser-Lyubeznik (HSL) numbers when reducing to characteristic p, and another on gathering open questions on various numerical invariants. Finally, another group also explored test ideal type constructions in mixed characteristic.

Wednesday morning began with a detailed overview by Tommaso de Fernex of a number of (very recent) ways of defining a multiplier ideal in the non-$\mathbb{Q}$-gorenstein setting. In particular, he highlighted various kinds of discrepancy one may use (such as the Mather discrepancy) appearing in recent work of T. de Fernex and R. Docampo, as well as L. Ein, S. Ishii, and M. Mustaţă. Next, Kevin Tucker gave a short talk on F-signature, ending with a great number of open questions. Lastly, Daniel Hernandez gave a short talk on F-pure thresholds, prompting a number of questions on various non-degeneracy conditions for polynomials in characteristic p. In the afternoon sessions, one of the groups attempted to prove a conjecture of K.-i. Watanabe and K.-i. Yoshida on minimal Hilbert-Kunz multiplicity in dimension seven. Another group reviewed what is known for global applications of test ideals, as well as clarifying the problems that need to be solved if one wishes to apply this theory in analogy with the characteristic zero picture. Finally, another group attempted to investigate a tight closure operation in characteristic zero constructed via resolution of singularities.

Thursday morning began with a talk by Kirti Joshi, where he reviewed his joint work with C. S. Rajan on ordinarity verses Frobenius splitting on also on ordinarity for K3 surfaces. Next, Craig Huneke gave a detailed overview of Hilbert-Kunz multiplicities. In the afternoon, we held a lively problem session moderated by Holger Brenner. In one of the afternoon sessions, Ian Aberbach showed how to answer one of the open questions presented earlier for F-signature, and together Craig Huneke and K.-i. Watanabe answered one of the open questions on a multiplicity bound for F-pure rings posed in the problem session by Karl Schwede. Another group worked on questions about F-pure thresholds and a final group continued to look at potential applications of test ideals to projective algebraic geometry.

Friday morning had three talks once more (the second two were shorter). Mel Hochster spoke on splitting conditions and the direct summand conjecture, particularly emphasizing various ways he has tried to approach the mixed characteristic case. Steffano Urbinati then spoke on the discreteness and non-rationality of jumping numbers for multiplier ideals in the non-$\mathbb{Q}$-gorenstein setting. Lastly, Wenliang Zhang spoke on the discreteness and rationality of F-jumping numbers and computing test ideals uniformly via a single alteration. In one of the afternoon groups, Florian Enescu reviewed a result of N. Shepherd-Baron on the upper-semicontinuity of the numbers in the Hilbert-Kunz function, and the group then attempted to show the upper-semicontinuity of Hilbert-Kunz multipilcity and lower-semicontinuity of the F-signature. Other groups continued their study of HSL numbers and F-pure thresholds.

By general consensus, the workshop as a whole was enjoyable and successful. Substantial partial progress was made on the focus questions for the workshop, and we expect a

number of papers and collaborations to appear in the coming months. Several participants said they particularly enjoyed the free-form style of the workshop and found the working groups particularly worthwhile. We are grateful to AIM for providing the necessary funding and a pleasant setting, and also to our participants – particularly the speakers – for sharing so generously of their time, knowledge, and ideas.