Symplectic four-manifolds through branched coverings

organized by
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Workshop Summary

Symplectic geometry, especially in dimension 4, lives at the fertile interface of algebraic geometry and 4-dimensional topology, drawing from and enriching both of those subjects. This workshop brought together researchers from these different fields to exchange ideas on key problems in the area. One of these key issues that we discussed is the question of whether the Bogomolov-Miyaoka-Yau (BMY) inequality [Bogomolov,Miyaoka,Yau1] between Chern numbers of complex surfaces extends to the symplectic setting. Another is the understanding of how branched covers interact with the geometry and topology of symplectic manifolds. The study of branched coverings is in turn closely tied to the study of singularities of curves, where an important goal is to understand how much of the delicate information about complex singularities can be derived from symplectic geometry or from topological tools such as Floer homology.

At the beginning of the week, we had a discussion of concrete problems pertaining to these broad issues, and then split into smaller groups designed to make progress. As the week went on, new problems emerged, and some of the research groups were to some degree reconfigured.

1. Morning talks

The goal of the morning talks was to introduce essential ideas and techniques from algebraic geometry, symplectic topology, and gauge theory that would facilitate discussions among people with diverse backgrounds.

1.1. Branched covers and other symplectic constructions Denis Auroux and Inanç Baykur discussed how smooth topological constructions can be used to build manifolds supporting symplectic structures. Auroux presented his theorem [Auroux] that all symplectic 4-manifolds have a symplectic branched covering map to the complex projective plane, and described the models for the singularities of that map. Then he explained how one can keep track of the branched covering data through the braid monodromy of the branching curve. Baykur gave an introduction to the geography problem in symplectic 4-manifolds and algebraic surfaces and described constructive tools that have been used to fill in various geographic regions. Then he gave explicit examples showing how these tools can be used to produce exotic symplectic manifolds and symplectic manifolds whose geography differs from that of algebraic surfaces.

1.2. Singularities of complex curves Jozef Bodnar, Karol Palka, Anatoly Libgober gave talks on questions and tools coming from algebraic geometry on singular planar curves. Bodnar gave an introduction to some of the open questions on rational cuspidal curves and some of the recent results and tools used. Palka discussed how the log minimal model program has been useful in algebraic geometry to prove [KorasPalka] the Coolidge-Nagata conjecture that all rational cuspidal curves are birationally equivalent to the line. Libgober discussed properties of fundamental groups of the complements of algebraic curves, explained the contributions of simple singularities, and properties and computations of the Alexander polynomial.

1.4. BMY. Roger Casals discussed the Bogomolov-Miyaoka-Yau inequality, giving a discussion of Miyaoka's proof. He broke down the proof into key lemmas, gave the algebro-geometric computations, and discussed which parts might be replicable in symplectic geometry.

1.4. Gauge-theoretic invariants. Eugene Gorsky and Daniel Ruberman talked about gauge-theoretic invariants in relation to the themes of the workshop. Gorsky gave an introduction to Heegaard Floer homology, especially focusing on the d-invariants, and how these invariant can be used to analyze algebro-geometric properties of curve/surface singularities in terms of topological properties, especially in the case of rational singularities. Ruberman gave an introduction to the Seiberg-Witten invariants of 4-manifolds and discussed the literature on how these invariants behave under branched covers.

1.5. Topology of branched covers. Alexandra Kjuchukova and Daniele Zuddas gave talks about the smooth topology of branched covers. In addition to basic topological properties of branched covers, Kjuchukova talked about using bridge trisections of singular surfaces to produce interesting decompositions of 4-manifolds obtained as irregular branched covers, and also how this could detect a potential counterexample to the Slice-Ribbon conjecture. Zuddas discussed his recent work on realizing manifolds with boundary as branched covers over a punctured S^4, and how this was used in his group's work to prove that all simply-connected 4-manifolds admit symplectic branched covers.

2. Afternoon problem sessions

The first afternoon began with a discussion of broadly motivating problems and specific questions that could be examined in smaller groups. On subsequent afternoons, we had short reports from each of the research groups on progress, obstacles, and connections to other research topics. Some of the problems, such as the distinction between the behavior of singular curves in the smooth and complex settings, and the existence of symplectic branched covers for arbitrary 4-manifolds, generated a good deal of interest during the week. Approaches to adapting the proof of the BMY inequality to the symplectic case were proposed, as were potential routes to finding symplectic counterexamples. Other initially promising research directions, such as the behavior of gauge theoretic invariants under branched covers, seemed too difficult to permit much progress.

A summary of activities in the different afternoon working groups:

2.1. Realize blowdowns/rational blowdown through branched covers, especially on $E(1)$. One of the major goals for the workshop was to describe operations on symplectic 4-manifolds through modifications of the branch sets describing them. While Luttinger surgery is well-understood in terms of twisting along certain annuli in the branch set, there was initially vocal skepticism about understanding blowdown/rational blowdown operations in this way, due to the fact that they necessarily would need to change the number of sheets of the covering and other fundamental aspects of the branching description. However, by Thursday, the group working on this was able to describe blow-down of -1-spheres in terms of a particular operation on the branch set, which is already exciting progress. Initial calculations on the first type of rational blow-down were also made, but the general case remains open. If this were generalized and systematized, then this would allow for the explicit description of the branch sets for many interesting symplectic manifolds whose basic algebraic topology lives outside of the complex/algebraic realm (e.g. below the Noether line).

2.2. Branched coverings by symplectic manifolds. The motivation was a question of Eliashberg roughly asking if every orientable 4-manifold admits a symplectic branched covering [**?**, Conjecture 6.2]. The group working on this topic took as a starting point recent work of Piergallini-Zuddas [piergallini-zuddas] on branched covers of $\mathbb{CP}^2$. By the end of the week they had shown that every simply-connected 4-manifold (and hence any manifold with finite fundamental groups) has a symplectic branched cover. While this answered the roughest form of Eliashberg's conjecture, there is a more precise statement that the branch locus be a symplectic surface together with additional homological data. The group will continue work on this topic, with the aim of expanding the results to arbitrary fundamental groups as well as satisfying Eliashberg's criteria.

2.3. Study the symplectic BMY conjecture. Many classical examples of complex surfaces on the BMY line (i.e. satisfying $c_1^2 = 3c_2$) were constructed by Hirzebruch [hirzebruch:singularities] as branched covers of $\mathbb{CP}^2$ branched along singular curves. The approach in one group was to try to modify those branch curves in a symplectic fashion in a controlled way that would change the Chern numbers appropriately and lead to a counterexample in the symplectic category. A related effort in another group sought to understand the symplectic nature of the Hirzebruch examples on the BMY and to search for Lagrangian tori from the perspective of a Lefschetz fibration. A Luttinger twist [luttinger:twist] along such a torus could change the Chern numbers, again providing an approach to finding a counterexample. This group was able to make significant progress, finding an explicit Lefschetz fibration and using this to find many Lagrangian tori in one of the simplest Hirzebruch example. It turns out that these Lagrangians were not homologically essential and thus do not work for this strategy. In fact, they found a general result that said that for topological reasons, there are no homologically essential tori in the ball quotients that lie on the BMY line.

2.4. Compare symplectic and algebraic planar curves. There are many restrictions on the collection of singularity types that an algebraic curve can have. Most of these restrictions were originally proven from highly algebro-geometric tools which do not directly apply to the symplectic category. However, more recently some topological techniques have been introduced which obstruct all smooth planar curves including both algebro-geometric and symplectic ones. One group studied examples of these restrictions, particularly focusing on the log BMY inequality. Since the log BMY inequality is only proven in the algebraic category, there could be counterexamples in the smooth or symplectic categories. This group came up with a collection of potential counterexamples. They were able to show that these first examples in fact can be obstructed topologically and therefore cannot be realized symplectically. This gives further evidence of the similarities between the symplectic and algebraic categories. Additionally, some progress was also made on methods of construction of symplectic configurations of low degree curves using real projective pseudocurves, though we currently lack an unobstructed example of interest to attempt to construct.

2.5. Study fundamental groups of the complements of configurations of surfaces. The fundamental group of the complement of a singular curve has some special properties [cohen-suciu:characteristic,dimca-papadima-suciu:jump] stemming from the fact that the complement is a quasi-projective variety. As explained in the morning talk by Libgober, these properties are most easily understood in the case when the singular curve is a union of smooth curves, even complex (projective) lines. In that case, it is useful to focus on the rich structure of the Alexander invariant (the homology of the universal abelian cover). The group working on this topic focused on examples [RubermanStarkston] of symplectic configurations, to see if the Alexander invariant would show them to be non-algebraic.

2.6. Algebraic cobordisms between algebraic links. The group discussed knots given by links of plane curve singularities and the relation between those knots arising from deformation of singularities. Participants considered a conjecture that one of these knots must embed in the Seifert surface of the other (currently, only a weaker statement involving concordance is known). One can consider the Milnor smoothings of the plane curve singularities in a deformation family. It is easy to show that one Milnor fiber is cobordant to a subsurface of another in the family, but it is surprisingly hard to show that the cobordism is a product cobordism. Such result would be very interesting because currently we do not have a method to distinguish a smooth cobordism between algebraic knots from a cobordism coming from a deformation of singular points. In particular, all known methods for obstructing an unfolding of one (topological) type of a singularity to another (topological) type, rely on obstructing smooth link cobordism. It is not known, whether the complex structure of the cobordism given by the deformation gives an additional restriction. Different possible approaches were discussed, such as modifying the classical Milnor fibrations tools or using open book decompositions and ideas from contact topology.

2.7. Branched covers and trisections. One new and promising perspective in smooth four-manifold topology is that of a trisection - a decomposition of the 4-manifold into three simple pieces. Many problems in four-manifolds can be reduced to finding “efficient” trisections, and previously Lambert-Cole, Meier, and Zupan have shown that the d-fold cyclic branched cover of a smooth degree d curve in $\mathbb{C}P^2$ admits such an efficient trisection. Using the work of Kjuchukova (as discussed in her talk), this group constructed bridge trisections for certain singular symplectic curves in $\mathbb{C}P^2$ and lifted these to efficient trisections of the branched covers. For instance, this gives new proofs that these manifolds admit perfect Morse functions. Since these techniques deal with more complicated branched covers (irregular dihedral covers), it is hopeful that this perspective will also give new tools for studying the branched cover descriptions of symplectic four-manifolds.

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