Conformal symplectic structures, contact topology, and foliations

organized by
Melanie Bertelson, Yakov Eliashberg, Gael Meigniez, and Emmy Murphy

Workshop Summary

Motivation and background

The workshop "Conformal Symplectic Structures, Contact Structures, and Foliations, in interaction" was organized by Mélanie Bertelson, Gael Meigniez, Emmy Murphy and Yakov Eliashberg in order to build on a recent progress on this area.

On a differential manifold of arbitrary even dimension, a *conformal symplectic structure* is a conformal class of nondegenerate differential forms of degree 2, admitting a local symplectic representative in a neighborhood of every point. The interest for this classical generalization of symplectic geometry has recently been renewed.

- An existence h-principle for conformal symplectic structures on closed manifolds has been obtained, as an application of two fundamental flexibility results of the last ten years in contact and symplectic geometry, namely the Borman-Eliashberg-Murphy h-principle for overtwisted contact structures [BEM] and the construction of symplectic structures on cobordisms in [EM];

- The conditions under which an integrable hyperplane field (foliation) can be deformed into a maximally nonintegrable one (contact structure) are well-known in dimension 3 after a classical work [ET] In larger dimensions, it appears that the condition more or less amounts to construct an exact conformal symplectic structure on the leaves of the foliation, as it was revealed in a recent work [BM] . To this aim, we brought together people from different areas of expertise: foliation theory, contact and symplectic topology.

The workshop

The goal of the workshop was to explore these recently discovered connections between conformal symplectic structures, contact topology, and the theory of codimension one foliations. The workshop was intended to review the recent progress related to the topic and to stimulate further research on the various new questions.

The recent existence results for conformal structures and exact leafwise conformal symplectic structures, as well as on deformation of leafwise conformal foliations into contact structures raise many interesting questions.

Because of the Covid 19 pandemy, the workshop could unfortunately not take place in person at AIM It was held online, using the Sococo software provided by the AIM.

The workshop had participants from North America, Europe and Japan The activities were happening mostly in the morning and early afternoon of Pacific time in America, the end of the afternoon and by night for the Europeans. It was especially difficult for Japanese participants: from the end of the night to the dawn.

All the participants understood and accepted the constraints, and appreciated the help of the software and of the AIM team to reconstitute some collective working ambiance. However, the general agreement of the participants that these conditions cannot compete with the traditional workshop format.

Following the AIM model, the workshop consisted of more or less formal lectures in the "mornings", and discussions, working groups, and open problem sessions in the "afternoons".

The first lecture on Monday, 15th, was given by Baptiste Chantraine, and titled "Lagrangian intersection in Conformally Symplectic Manifolds". He first reviewed the background on Conformal Symplectic structures (a.k.a. Locally Conformal Symplectic structures, or lcs structures). Then, he introduced a recent work in collaboration with E. Murphy. They get a version, in the conformal symplectic framework, of the classical Laudenbach-Sikorav theorem for the persistence of intersections of Lagrangian submanifolds in cotangent bundles.

In the second Monday talk, Mélanie Bertelson explained the "Construction of Conformal Symplectic structures" (joint with G. Meigniez) using the Borman-Eliashberg-Murphy h-principle for overtwisted contact structures, and the Eliashberg-Murphy symplectization of cobordisms.

On Wednesday "morning", Gael Meigniez continued Bertelson's lecture with "From foliations to contact structures in high dimensions" (joint with M. Bertelson): how the construction can be applied to the leaves of a foliation, and why it provides a linear deformation of the foliation into a contact structure. After the talk, a discussion raised about the difficulties to carry the construction on closed manifolds. These difficulties seem to be connected with the question of the existence or not of a version of Gromov's compactness for pseudo-holomorphic curves in this foliated and/or conformal framework.

The second Wednesday talk was by Emmy Murphy. After she explained basics of local conformal structures, she explained the construction of lcs whose Lie form is of rank one, using the h-principle for overtwisted contact structures, and the construction of symplectic cobordisms from [EM].

On Thursday, the second talk was by Yasha Eliashberg who gave a more detailed account of the construction of symplectic cobordisms. The discussion after the talk carried on several open questions of naturality or uniqueness up to isotopy of the construction.

The first talk of Thursday was given by Jonathan Zung: "Contact approximations of Foliations and Diffusion processes". Applying some methods of diffusion (or dually Brownian motion) along the leaves of a foliation in a 3-manifold, he obtains an important refinement of the Eliashberg-Thurston theorem: if the foliation admits no transverse invariant measure, then the Reeb vector field of the approximating contact form can be made transverse to the foliation. Several participants are specially interested in the question if his methods and result admit some generalization in higher dimensions.

The first talk of Friday was about the much connected subject of genuine symplectic structures on foliations: Yoshi Mitsumatsu explained how to construct such objects, "Lefschetz fibration on the Milnor fibers of simple elliptic and cusp singularities" (joint with Naohiko Kasuya, Hiroki Kodama and Atsuhide Mori)

Finally, Lauran Toussaint explained obstructions to approach foliations by contact structures in high dimensions, in several senses; his lecture was titled "Convergence of contact structures to (conformal symplectic) foliations" (joint with Fran Presas).

The afternoon working groups were devoted to the following topics:

- What is the correct notion of overtwistedness for conformal symplectic structures? Is there a full parametric h-principle for these structures?
- Making the first steps of the theory of tight conformal symplectic structures;
- Exploration of rigidity phenomena in conformal symplectic topology;
- Construction of codimension one leafwise conformal symplectic foliations;
- Exploration of the dynamics of Liouville vector fields in the conformal (or not) symplectic setting.

The working groups made some progress which allowed participants of the workshop to continue their work in these directions after the workshop.

What became clear is that it is important to properly setup foundations of this new emerging area. One of the workshop participants, Baptiste Chantraine took upon himself an organization of a regular international working group to complete this task. The group has been meeting online every two weeks since April in the Sococo space and made a significant progress.

Bibliography
[BM] M. Bertelson and G. Meigniez, *Deforming foliations into contact structures in all dimensions,* in preparation.

[BEM] M. S. Borman, Y. Eliashberg, E. Murphy, *Existence and Classification of Overtwisted Contact Structures in all Dimensions,* Acta Math. 215 (2015), 281–361.

[EM] Y. Eliashberg and E. Murphy, Making cobordisms symplectic. ArXiv : arXiv:1504.06312.

[ET] Y. Eliashberg and W. P. Thurston, *Confoliations*, University Lectures 13, American Mathematical Society, Providence, RI (1998).

[BM] M. Bertelson and G. Meigniez, *Deforming foliations into contact structures in all dimensions,* in preparation.