Steklov eigenproblems

organized by
Oscar Bruno, Michael Levitin, Nilima Nigam, and Iosif Polterovich

Workshop Summary

The major area of interest for the workshop participants was the study of the *Steklov eigenvalue problem*. Let Ω be a compact Riemannian manifold with boundary $\partial\Omega$. Consider the following eigenvalue problem

$$\Delta_g u = 0 \text{ in } \Omega, \quad \partial_\nu u = \sigma u \text{ on } \partial\Omega,$$

where ∂_ν is the outward normal derivative and Δ_g is the Laplace-Beltrami operator. If Ω is a Euclidean domain, the operator Δ_g is the usual Laplacian. The Steklov spectrum coincides with the spectrum of the Dirichlet-to-Neumann operator on Ω, and the boundary traces $\phi_\lambda = u_\lambda|_{\partial\Omega}$ of the Steklov eigenfunctions are precisely the eigenfunctions of the Dirichlet-to-Neumann map. Given the importance of the Dirichlet-to-Neumann map in applications ranging from medical imaging to the computation of PDE in unbounded domains, progress in our area of interest could have wide mathematical impact.

Our goals were to combine deep theoretical analysis with highly accurate and efficient numerical methods in order to study various geometric features of the Steklov eigenvalues and eigenfunctions. The questions of interest are deeply challenging because they are at the forefront of research in both numerical analysis and spectral geometry. This workshop was the first opportunity to gather together "analytic" and numerical communities working on Steklov-type problems, brainstorm, and share expertise across fields. We expect that advanced analytic techniques would help creating "right" numerical methods, which in turn should lead to discoveries of new phenomena and to better geometric and analytic intuition.

Our main aims were to make progress on the following four related topics:

(1) Nodal geometry of Steklov eigenfunctions.
(2) Shape optimization of Steklov eigenvalues.
(3) Steklov type problems for Maxwell and Lamé operators.
(4) Spectral asymptotics for singular Steklov type problems

The mornings on each of the five days were devoted to lectures, usually one with more analytic flavour, and one computational (though, as it turned out, many talks successfully combined both aspects). On Monday, Leonid Parnovski spoke on eigenvalue asymptotics for the Steklov problem on domains with corners, and Oscar Bruno gave a lecture on Steklov eigenvalue and eigenfunction computations using integral operators. On Tuesday, Mikhail Karpukhin presented a talk on Steklov problems for differential forms, with particular applications to Maxwell operators, and Peter Monk spoke about the use of Steklov operators in inverse scattering problems. The talks on Wednesday were given by Jeffrey Galkowski on microlocal techniques and their applications to the study of Steklov eigenfunctions, and by

Mihai Putinar on very interesting links between the Steklov and Neumann-Poincaré operators.

On Thursday we had the talks of Ailana Fraser on minimal surfaces and optimal isoperimetric inequalities for Steklov eigenvalues, and by Braxton Osting on numerical eigenvalue optimisation. Finally, on Friday, after Fioralba Cakoni's talk on transmission eigenvalue problems, we had three shorter talks by PhD students: Jean Legacé on Steklov eigenvalues of higher-dimensional cuboids, Emmanuel Garza on high-accuracy computation of modes in waveguides, and by Sebastian Dominguez on Bayesian optimization of eigenvalues.

The afternoons were devoted to group work on specific problems and general discussions. Of more than 25 problems identified on Monday, about 8 or 10 were addressed, with groups sometimes joining together. Eventually, we made concrete progress in formulating 6 main problems, and established connections to continue working on these:

- Steklov problems for Helmholtz, Lamé and Maxwell in two and three dimensions. The Maxwell problem lead to a remarkable breakthrough in the meeting, where three distinct formulations (two by numerical analysts, and one by a geometer) were found to have deep connections. A suitable formulation for investigation by a larger group has been proposed.
- Neumann-Poincaré operators and their spectra. Neumann-Poincaré operators have a rich history and their spectra are related to the Steklov spectra. It is an interesting question to explore this link in the presence of domain singularities.
- The intriguing question regarding the density of nodal sets for an ellipse. High-accuracy calculations suggest that there are sets of non-zero measure in the interior where the high frequency eigenfunctions do not change sign; proving this result would be a fascinating challenge.
- Is there an analogue for the Steklov problem of the Polya-Szego conjecture on polygons, i.e. is true that regular polygons maximize the appropriately normalized first Steklov eigenvalue

  among all polygons with a given number of sides? How might one prove this? A surprising hurdle was the lack of a shape derivative for this setting.
- Consider the mixed Steklov-Neumann or Steklov-Dirichlet problem, where the Steklov portion of the boundary is of fixed fraction. We seek to optimize a given part of the spectrum (say the first non-zero eigenvalue). What is the optimal configuration of zero Neumann data? This is akin to the 'narrow escape' problem.
- Can the tools of microlocal analysis be exploited to yield more accurate approximation-theoretic strategies for the Steklov problem?