Stability and moduli spaces

organized by
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Workshop Summary

Workshop overview

This workshop was devoted to recent developments in the construction and study of moduli spaces in algebraic geometry arising from invariant theory, birational geometry, differential geometry, and category theory. Specifically, the first goal of the workshop was to synthesize the different notions of stability resulting from four different points of view, namely

(1) Geometric Invariant Theory (GIT) stability,
(2) Kollár–Shepherd-Barron-Alexeev (KSBA) stability,
(3) K-stability, and
(4) Bridgeland stability.

The second goal of the workshop was to provide specialists in each of these areas with a chance to learn about the other threads from leading experts and to seed future interactions across the boundaries of each. The third goal of the workshop was to identify fundamental open problems in the broad area of moduli spaces, and to generate fresh perspectives on these problems using ideas from one or a combination of the four points of view mentioned above. Towards these goals, we brought together experts in Geometric Invariant Theory, algebraic stacks, moduli of curves, moduli of surfaces, minimal model program, Bridgeland stability, Kähler-Einstein geometry, and K-stability for a week long workshop at the American Institute of Mathematics.

Following the AIM model, we scheduled two formal talks each morning, which provided an overview of the significant recent developments in each of the four threads, emphasizing connections with other threads, open problems, and possible approaches to addressing these problems. During the afternoon on Monday and half of the afternoon on Tuesday, we scheduled four parallel discussion sessions, each devoted to one of the four threads. Moderated by young researchers, these informal sessions gave everyone an opportunity to get acquainted with the basic notions of unfamiliar areas in an informal setting in which little background knowledge was assumed and participants felt free to clarify even the most basic issues.

During the second half of the afternoon on Tuesday, Gavril Farkas moderated a problem session. Equipped with a working knowledge of the basic techniques and challenges in each thread, the participants engaged in a lively discussion and generated a list of about 20 problems. Out of these, we compiled a structured list of 10 problems, which formed the basis of the working groups that met during the afternoon on Wednesday, Thursday, and Friday. On Friday afternoon, each working groups reported on its progress. The second section of this report gives a summary of these reports.

The workshop as a whole was productive and enjoyable (despite the damp weather). The working groups made substantial progress on the questions they considered. To our satisfaction, many participants reported to us that they had acquired a valuable introduction to previously unfamiliar areas and entered into new collaborations with researchers in these fields. We expect these collaborations to continue in 2017 and look forward to hearing their results in the coming months.

Summary of working group activities

Working group on "The asymptotic Chow stability of cubic surfaces" (presented by Zhiyuan Li)

This group sought to establish whether GIT stability coincides with asymptotic Chow stability for cubic surfaces, motivated by a question of Laza and Odaka relating K-stability, GIT stability and asymptotic Chow stability. Using Kempf's reduction of G-stability for a general G to a maximal torus and Ono's criteria to test Chow semi-stability with respect to a torus action, the group is very likely to establish the asymptotic Chow stability of the cubic surface in $\mathbb{P}^3$ defined by the equation $xyz = w^3$.

The group made the following observation and posed an accompanying problem: Let G be a finite group and X a G-variety. It is known that if X is K-stable, then so is X/G at least in the case when X^G, the set of fixed points, is of codimension at least two. Is the analogous statement true for asymptotic Chow stability for toric varieties?

Working group on "The Donaldson–Futaki invariant and the MMP" (presented by Giulio Codoni)

This group sought to understand the effect on the Donaldson–Futaki invariant of running the minimal model program on the test configuration. The motivation was to reduce the set of test configurations to check to a more manageable set, using the results from birational geometry.

Let (X, L) be a polarized variety, $(\mathcal{X}, \mathcal{L})$ a test configuration for (X, L), and $\mathcal{X}^{\mathrm{lc}} \to \mathcal{X}$ a log canonical model of $(\mathcal{X}, \mathcal{X}_0)$. Under the assumption that $\mathcal{X}$ is $\mathbb{Q}$-Gorenstein, the group found a relationship between $\mathrm{DF}(\mathcal{X}^{\mathrm{lc}}, \mathcal{L}^{\mathrm{lc}})$ and $\mathrm{DF}(\mathcal{X}, \mathcal{L})$ in terms of an intersection number on $\mathcal{X}^{\mathrm{lc}}$. They explained the challenge in relaxing the $\mathbb{Q}$-Gorenstein hypothesis.

Working group on "A GIT compactification of the Hurwitz space" (presented by Maksym Fedorchuk).

Denote by $\overline{M}_g(\mathbb{P}^1, d)$ the Kontsevich compactification of the moduli space of maps of degree d from genus g curves to $\mathbb{P}^1$. This group aimed to analyze the GIT quotient $\overline{M}_g(\mathbb{P}^1, d)/\!/\,\mathrm{SL}(2)$. Their motivation was to find a compactification of the classical Hurwitz space that is more suitable for divisor computations than the standard compactification using admissible covers.

By considering the branch divisor morphism $\overline{M}_g(\mathbb{P}^1, d) \to \mathbb{P}^{2g+2d-2}$, they showed that for an appropriate choice of linearization, a point in $\overline{M}_g(\mathbb{P}^1, d)$ ought to be (semi)-stable if and only if its image in $\mathbb{P}^{2g+2d-2}$ is (semi)-stable. They raised the interesting problem of providing a good modular description of the resulting GIT quotient.

Working group on "Bridgeland stability conditions on degenerate varieties" (presented by Xiaolei Zhao)

This group aimed to find Bridgeland stability conditions on singular and reducible varieties. Their motivation was to use stability conditions on degenerate varieties along with a smoothing argument to address the central open problem of constructing Bridgeland stability conditions on (smooth) quintic threefolds. After settling the cases of reducible curves and reducible surfaces in $\mathbb{P}^3$ given by unions of k-planes for small values of k, the group approached the more general problem using a categorical resolution of $D^b(\mathrm{Coh}X)$ for a singular X provided by Kuznetsov and Lunts. They explained the difficulties in this approach and the need for finding a more minimal categorical resolution.