Stability in mirror symmetry

organized by
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Workshop Summary

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Organizers: Tristan Collins and Jason D. Lotay

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Introduction.

This workshop focused primarily on two geometric objects which both play a key role in the study of mirror symmetry: special Lagrangian (sLag) submanifolds and deformed Hermitian-Yang–Mills (dHYM) connections. The sLag submanifolds and dHYM connections are expected to be related by mirror symmetry and their existence is conjectured to be governed by a stability condition. As a consequence, stability conditions are expected to play a crucial role in understanding the behaviour of geometric flows which have sLags and dHYM connections as critical points: namely, the Lagrangian mean curvature flow (LMCF) and the line bundle mean curvature flow (for rank 1 dHYM connections) respectively.

The workshop brought together a diverse range of researchers, both specialist in the area and in interrelated topics, and there was a good mix of junior researchers (graduate students and postdocs) and established researchers. As a consequence, there was a great interplay between the participants, each bringing their own interests, knowledge and ideas, and there was a lot of lively discussion.

Summary.

The main topics discussed in the workshop were as follows.

- Lagrangian mean curvature flow: relation to Bridgeland stability conditions, Fukaya categories and J-holomorphic curves; singularity formation and long-time existence; soliton solutions.
- Deformed Hermitian-Yang–Mills connections: relation to Geometric Invariant Theory and stability; existence results; higher rank.
- Mirror symmetry: special Lagrangians in Landau-Ginzburg models; toric mirror symmetry and sheaves; proof of mirror symmetry in examples.

Each day, as per the usual AIM format, the workshop consisted of two talks followed by problem sessions, as well as allowing for time for informal discussion.

Talks.
The talks served both to provide background and surveys, as well as give more specific talks on results, including showcasing of very recent work by junior participants.

Felix Schulze gave a survey on singularity formation in LMCF. Felix described recent progress in our understanding in standard mean curvature flow, specific results pertinent to LMCF, and described some important open problems.

Adam Jacob gave a survey on the dHYM equation. Adam described the origin and properties of the equation, its relation to mirror symmetry and stability, progress on existence of solutions, the line bundle mean curvature flow, and some key open problems.

Dominic Joyce discussed his revised version of the Thomas–Yau conjecture and LMCF. Dominic described how to formulate his conjecture in terms of Bridgeland stability and Fukaya categories, and his programme for proving the conjecture using LMCF, detailing the key singularity models for consideration and some important examples.

Tristan Collins described the relationship between GIT and the study of the dHYM equation. He discussed the parallels between dHYM connections and sLags, the links to Kempf–Ness functionals and geodesics in certain non-positively curved spaces, as well as stability conditions.

Chris Woodward gave a more symplectic perspective on geometric flows, sLags, surgery and links to Floer cohomology.

Eric Zaslow gave a different perspective on mirror symmetry for toric varieties using the theory of constructible sheaves.

Fabian Haiden gave an introductory lecture on the basics of triangulated categories and Bridgeland stability conditions.

Jingyi Chen gave a summary of a variety of analytic results which helped to develop a compactness theory for the space of Hamiltonian stationary Lagrangian submanifolds, particularly in the case of 2-dimensional submanifolds and especially tori.

Chris Evans described work from his PhD, which he was about to finish, on LMCF in positive Kähler–Einstein manifolds. In particular, under a symmetry assumption, he described a general long-time existence and convergence result for LMCF with surgeries in $\mathbb{CP}^2$, showing that embedded monotone Lagrangian tori (including Chekanov tori) converge to minimal Clifford tori.

Yu-Shen Lin focused on joint work with Tristan Collins and Adam Jacob which had just been released that day proving SYZ mirror symmetry for del Pezzo / rational elliptic surface pairs.

Problem sessions.
The problem sessions went very well, with participants exploring the different topics, having engaging discussions, and forging new academic relationships. The main problems discussed were as follows.

(1) **Is the Lawlor neck a generic singularity of LMCF?** The group discussed the appropriate notion of genericity for the LMCF, comparing and contrasting with

recent progress in the usual mean curvature flow. Links to various notions of stability were also discussed. An approach to showing that a Lawlor neck singularity is stable under small perturbations of the initial condition was proposed by D. Joyce, involving the use of analysis on manifolds with corners. An approach to showing that, under generic choices of initial conditions, one could remove all singularities except those modelled on Lawlor necks was put forward by F. Schulze, in the spirit of the usual mean curvature flow.

(2) **Can one construct sLags in Landau–Ginzburg models, and what is their relation to dHYM connections on the mirror Fano?** In this group an interesting nonlinear PDE problem connecting special Lagrangians with microlocal sheaves and Legendrian fronts was described by E. Zaslow. An approach to this problem by gluing methods was suggested. There was further discussion on the connections between calibrated Lagrangians and stability conditions for Landau-Ginzburg models. It was pointed out that examples on Fano surfaces suggest that the notion of a Bridgeland stability condition may need to be modified in order to predict the existence of special Lagrangian submanifolds. Ideas for how to modify the notion of a stability condition were discussed.

(3) **What is the correct notion of the dHYM equation in higher rank? Can we construct solutions to this equation, and what are the mirror sLags?** This group found a natural higher-rank version of the dHYM equation. It was suggested that one could construct solutions of the dHYM equation on non-split extensions of the tangent bundle and the structure sheaf on a Calabi–Yau surface. There was further discussion about the possible geometric implications for sLags on the mirror manifold.

(4) **How can one use J-holomorphic curves to understand the formation of singularities in LMCF?** Following D. Joyce's talk about the Thomas–Yau conjecture, participants discussed how the techniques of J-holomorphic curves and especially the notion of obstructedness in the Fukaya category, should play a role in the study of LMCF. Moreover, the role of J-holomorphic curves in understanding singularities modelled on Lawlor necks and Joyce–Lee–Tsui translators was clarified.

(5) **Are the Joyce–Lee–Tsui translators unique (under suitable conditions) as eternal solutions to LMCF?** The group (led by J. Lotay) proved uniqueness of the positive time blow-down for appropriate eternal solutions of LMCF with the same asymptotics as Joyce–Lee–Tsui translators. W.-B. Su described the geometry of these translators in detail. As a consequence, a promising method of continuity approach to the uniqueness was proposed, building on work of Lotay–Neves, Joyce–Imagi–Oliveira dos Santos and Lambert–Lotay–Schulze. Several members of the group plan to follow this proposed approach.

(6) **Can one use the relationship between GIT and LMCF to obtain convergence results for the flow?** There was some very interesting discussion on how ideas from Geometric Invariant Theory might be used to study the convergence of the LMCF. Analogues with the Kähler–Ricci flow, particularly work of Phong–Sturm, were discussed. It was suggested that these ideas might be particularly useful in the case of almost calibrated, strictly stable or semi-stable Lagrangians. The members of the group suggested several future directions which they plan to pursue.

In summary, important progress was made towards each of these problems and it seems very likely that these discussions will yield concrete progress on the connections between the dHYM/sLag equations and stability conditions.

Conclusion.

The organizers were extremely pleased with the results of the workshop, which greatly exceeded their expectations. As well as the participants learning a lot about the area, there was tangible progress on numerous fronts, new collaborations were developed, and the workshop is certain to lead to significant results for the field in the near future.

The organizers are grateful to the AIM staff for providing an extraordinary level of support for this very successful online workshop.