Special holonomy and branes

organized by
Sergei Gukov, Rafe Mazzeo, and Sakura Schafer-Nameki

Workshop Summary

The original intent of this workshop was to focus on a set of topics of substantial interest and importance at the intersection of mathematics and physics, namely the investigation of phenomena related to manifolds of special holonomy, their calibrated submanifolds, and the various constructions, from geometry or physics or elsewhere, that lead to these objects. There is an ambitious and now rather long-standing collaboration devoted to similar themes funded by the Simons Foundation (the Simons Collaboration on Special Holonomy in Geometry, Analysis, and Physics), and in the initial planning for this AIM meeting, an attempt was made to stake out some territory that was not in the main area of focus of this other group. However, as the planning evolved, it became clear that it would be highly desirable to attract some key participants who were part of that Simons Collaboration group, so a decision was made to make this a joint meeting, including both the initial AIM invitees and some subset of the main Simons collaboration members.

The meeting took place in an online format, using Sococo and Zoom, during the week of October 26-30, 2020. The chosen format was a bit different than had been recommended by AIM; namely, the first talk of the day was scheduled at 8AM CA time, but the main discussion problem session was held later in the morning and the second talk of the day was scheduled at 1PM. The hope was that this schedule might better accommodate the European and Asian participants.

The two talks on each of the first two days had an unusual format; each was a 90 minute talk, but split between two speakers, one who would present a more physics-oriented talk and the other a more mathematics-oriented talk. Each speaker had 45 minutes, and we asked them to coordinate carefully beforehand. For the rest of the week there were two more standard hour-long talks on Wednesday and one each on Thursday and Friday.

The speakers were:

Monday:
- Lara Anderson/Andreas Braun (morning)
- Jason Lotay/David Morrrison (afternoon)

Tuesday:
- Laura Fredrickson/Du Pei (morning)
- Andriy Haydys/James Sparks (afternoon)

Wednesday: Lorenzo Foscolo (morning)
- Rodrigo Barboso (afternoon)

Thursday:

Craig Lawrie (morning)

Friday:

Thomas Walpuski (morning)

Problem Sessions:

Starting on Monday after the morning talk, the group began to list problems of interest and by that afternoon had settled on six working groups. These initial problems and the working groups are as follows:

Problem List:

(1) ADE singularities and gauge/Higgs moduli spaces: possible theorems?
(2) Metrics on local models. Complete? Reconstruct from physics?
(3) Non-abelian configurations
(4) Understand physics of poles of connections A and Higgs fields ϕ
(5) Quantum effects: instantons, calibrations
(6) Quantum moduli spaces of, e.g., G_2 holonomy moduli spaces. Only understood in a very few cases.
(7) Determine/classify local models of codimension 4 and codimension 7 singularities in G_2 spaces
(8) Construct compact G_2 spaces with codimension 7 singularities.
(9) (Gukov) What is the physics behind ALG* spaces? How do they arise in physics? Comments: (Cherkis) – moduli of conformal $4dN = 2on R^3 \times S^1$ with four-dimensional moduli space.
(10) (Foscolo): Are there gluing theorems for $K3$ metrics using ALG* spaces? (or degenerations of $K3$ metrics to ALG* spaces?). See recent work of Chen, Viaclovsky, Zhang.
(11) (Schafer-Nameki) How can one characterize the kinds of singularities corresponding to physical CFTs? The CY_3 case, are the singularities canonical? Comment: (Joyce). Is the physical moduli space of $4dN = 1$ theories the same as the critical points of a functional defined on a G_2 moduli space?
(12) (Grassi): Kawamata log terminal $(K_i > 1), logcanonical(K_i \geq 1)$. What is the physical meaning? Many things known about deformations of terminal singularities. Much less known about deformations of canonical singularities.
(13) (Joyce) What are the mathematical consequences of CFTs, e.g. for stability conditions, categories?
(14) (Barbosa): Is there a 3-dimensional analogue of A/B models? Comment: probably not since no useful micro description of M theory exists.
(15) (Anderson) Canonical and Beyond Boundary beyond using Hitchin system, virtual dimension?
(16) Metrics on canonical/terminal CY_3 singularities. $KLT-CY$. See paper of Eyssideux, Guedj, Zeriahi on Ricci flat metrics on projective N-folds with KLT singularities. Comments: (Sun) If singularity is smoothable (i.e. deformable), then there is a unique tangent cone. The case of terminal singularities is technically difficult but expected to work.

(17) (Gauntlett) In AdS/CFT, conical metrics (which do not always exist). M-theory and 5D CFTs. Metrics might be different. G-K (Gauntlett-Kim) geometries. To be discussed by Sparks tomorrow.

Working Groups:

(1) Higgs bundles and local Models [could be several groups]
  1.1 G2/Spin7/CY4 compactifications Moduli space of Higgs bundles, Classification of local singularities Higgs bundles -¿ compact singular special holonomy spaces.
  1.2 Branes and Higgs bundles: GL Mirror Symmetry for wild Higgs bundles Higgs bundles and F-theory
  1.3 M5-brane correspondences: M6 on (c0)associatives, $\chi(M)$ Duality between 1.1 and wrapped M5 brane backgrounds
(2) Physics/Geometry of ALG*
(3) Singularities in special holonomy and SCFTs Geometric criteria; implications for geometry
(4) Metric aspects on singular G2 manifolds WP^{pppqq},
(5) GK-Geometries and Holography Cone Geometries
(6) Enumerative questions: Cayley, co-associatives

Report on activities of working groups

(1) Higgs Bundles for Calabi-Yau Four-Folds, $G2$-Manifolds, and Spin(7)-Manifolds

String dualities relate Higgs bundle vacua that arise in F-theory, M-theory, and the heterotic string. We considered the BHV (Beasley-Heckman-Vafa) system over a Kahler surface, derived from F-theory constructions, and the PW (Pantev-Wijnholt) system over a three-manifold, which comes from M-theory. The BHV system is naturally holomorphic in its formulation, whereas the PW system lacks this structure. In certain situations, these systems are expected to be dual, and we are interested in finding the analog of holomorphicity on the PW side of the duality. In particular, the data on the BHV side is contained in a spectral equation written in terms of holomorphic variables; do the PW spectral equations, written in terms of real variables, have corresponding extra structure? This question may be approached from the heterotic perspective, where both the BHV and PW systems appear natural.

We began by comparing two formulations of the PW system. The first formulation is in terms of 7D BPS equations modulo unitary gauge transformations. This is the setting in which physics analysis has primarily been carried out and where Higgs bundle vacua map to G2-geometry. The spectrum, chiral index, and geometry are clearly accessible from this perspective.

The second formulation is in terms of complex flat connections modulo complex gauge transformations. This perspective is closer to the holomorphic formulation on the BHV side, but it is less obvious how to carry out the analysis of the physical and geometrical data. However, it is clear that quantities such as the chiral index must be encoded in gauge-invariants of the complex flat connection (with respect to complex gauge transformations). To this end, we analysed how Wilson line and Wilson surface operators, the gauge-invariant data of the flat connection, may be used to compute

the spectrum and chiral index in the case of an abelian gauge group, following the logic of Pantev-Wijnholt.

(2) ALG and ALG* spaces. The participants in this group spent a considerable amount of time discussing the newest advances. Chief amongst these was a report by Gao Chen on his work in progress on Torelli theorems for ALG and ALG* spaces. The Torelli parameters do turn out to be the integrals of the three Kaehler forms over the generators of the second homology, as usual, but for cases where the underlying base of the fibration is conic with cone angle bigger than pi, there are extra technicalities due to the existence of harmonic 2-forms which decay at a relatively slow rate. Parts of this are related to ongoing work by Laura Fredrickson and Rafe Mazzeo and their collaborators; the thrust of that work is a verification of the 'modularity conjecture of Boalch, which asserts that all of these 4-dimensional hyperKaehler metrics should arise as Hitchin moduli spaces. The work of Fredrickson et al., has verified this in one special case (ALG spaces of D4 type). This led to some thoughts on how to reconcile the different perspectives. These spaces have appeared prominently in both older and more recent work of Sergey Cherkis, and he and Sergei Gukov began what has turned into a long conversation about the physics aspects, which had been one of the key motivations for this group. In particular, Gukov and Cherkis continued collaborating on the physics and brane realizations of ALG* metrics past the dates of the AIM workshop. Among different types of metrics, these spaces are more subtle because the most natural candidates for the relevant physical systems are not conformal in four dimensions. This leads to new phenomena and makes them more delicate compared to the other types of metrics that arise from conformal four-dimensional theories (compactified on a circle). Currently, Gukov and Cherkis are working on a concrete problem of realizing ALG* metrics in a slightly different way by using branes in type II string theory. Hopefully, this will lead to a new interplay between geometry and physics.

(3) Special Holonomy and Superconformal field theories:

Much progress in recent years has been made in the classification and analysis of superconformal field theories (SCFTs) in 5d and 6d. Such theories are intrinsically strongly coupled and have a realization in terms of canonical three-fold singularities. One aspect of the discussion in this group related to terminal singularities and their interpretation in terms of SCFTs.

Another more expansive discussion focused on the question of realizing 4d SCFTs with N=1 supersymmetry from either F-theory on elliptic Calabi-Yau four-folds or M-theory on G2 holonomy spaces. A strategy that provides a geometric realization of the recent works of Razamat and Zafrir was developed. The latter consider $4dN = 1$ SCFTs obtained by compactification of 6d SCFTs on Riemann surfaces. A geometric counterpart in F-theory is provided by taking the elliptic Calabi-Yau three-fold that realizes the 6d SCFT and fibering this over the surface. The challenge however is to control for quantum corrections that influence the superpotential of such minimally supersymmetric models.

The AIM workshop initiated discussions on this topic and in the meantime we have a concrete proposal how to construct candidate geometries.

(4) Metrics and New Constructions of Special Holonomy Spaces.

i). Various approaches to constructing/proving existence of various conjectural $G2$-holonomy metrics that are motivated by physics considerations were discussed. This included the co-associative adiabatic fibrations picture and local versions of the related Joyce-Karigiannis construction which utilises properties of a harmonic 1-form on the base of such fibrations. The possibly simplifying features of 1-forms with high order zeroes were briefly discussed/proposed. (Acharya, Bryant, Foscolo, Goette, Joyce, Lotay, Salamon)

ii). HyperKahler metrics were discussed in various contexts. The Bielawski-Foscolo construction (cf. Foscolos presentation) was reviewed with a view to clarifying its physical interpretation. The possibility that it could be used in a gluing construction to construct compact hyperKahler manifolds was discussed. Compact hyperKahler orbifolds arising from generalised Kummer constructions were also discussed and progress was made on producing new examples arising from compact Lie groups and root systems. (Acharya, Bryant, Foscolo).

Overall the participants found this discussion space engaging and quite productive.

(5) GK geometry and singular geometry for AdS/CFT

Discussion included several clarifications concerning the talk given by James Sparks were made. Discussions between Sparks and Gauntlett led to the exciting possibility of a new construction associated with branes wrapping certain calibrated cycles. A concrete co-homegeneity one ansatz was developed and it is hopeful that this will lead to the construction of new explicit solutions if the ODEs suitably behave themselves. Such solutions will be dual to new supersymmetric conformal field theories, via the AdS/CFT correspondence, and if successful will very likely open up new problems to work on.

(6) (Enumerative invariants session) Some topics we discussed:

(a) Enumerative invariants counting compact coassociative 4-folds C in a $G2$ manifold X. We proposed it might be a good idea to count C with $b2+(C)=0$. There are two reasons for this:

(i) Such C are rigid by McLean's Theorem, so you can count as a number.

(ii) There is a singularity of coassociative 4-folds, of real codimension 1 in the moduli spaces, based on Harvey-Lawson's $SU(2)$-invariant coassociatives in$R7 = C2 + R3$. This involves a coassociative with the local topology $O(1) - - > CP1$ collapsing to a cone. It is only possible if C contains an $S2$ with self-intersection 1, which does not happen when $b2+(C)=0$. So the assumption excludes this kind of singular behaviour, which might otherwise break deformation-invariance of the count.

(b) Wall-crossing phenomena for coassociatives in which two coassociatives $C1, C2$ intersecting in $S1$ generate a third by connected sum along $S1$, gluing in a family of Lawlor necks in $C3$. If $C2 has b3(C2) = 1 and H3(X) - - > H3(C2) surjective, so C2$ exists only in a real hyperplane of $G2$ structures with $[\phi|_{C_2}] = 0 in H3(C2)$, then this would give an interesting wall-crossing for the count of coassociatives $C1$ as you cross this real wall in the family of closed $G2$ structures.

Overall the workshop was successful in that it generated various concrete projects and led to folllow-up discussions. The format also worked well, and in fact will be taken up in future virtual meetings of the Simons collaboration as well.