Sheaves and modular representations of reductive groups

organized by
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Workshop Summary

This workshop took place at a time of intense activity among researchers working on geometric methods in modular representation theory. One of the goals of the workshop was to bring together experts in the algebraic aspects of the representation theory of reductive groups with those working with more geometric methods, with the aim of fostering an interchange of ideas that might lead to new progress.

Each day began with lectures on topics of current interest to the representation theory community. On Monday afternoon, Anthony Henderson led a moderated problem session. The input of the participants during that session was used to formulate a list of questions or topics for working groups. Most of the afternoons from Tuesday on, along with part of Friday morning, were devoted to the working groups. The following paragraphs below discuss some of the highlights of the week's working groups.

Positivity questions on flag varieties

On Monday and Tuesday morning, Brian Parshall, Leonard Scott, and Hangkyung Ko gave lectures around the theme of the Parshall–Scott "forced grading" program. One remarkable result to emerge from this program is that for a reductive group G in characteristic $p > 2h-2$ (where h is the Coxeter number), one can construct new highest-weight category related to $\mathrm{Rep}(G)$ by a kind of "associated graded of a filtration" construction. On Monday, Pramod Achar explained how a version of this result (combined with an integral generalization of a theorem of Soergel) implies that the indecomposable parity sheaves on the flag variety (with coefficients in characteristic p) are perverse. This means that the derived category of the flag variety is controlled by a *positively graded* dg-algebra. Progress on this positivity question was one of the aims put forward in the original workshop proposal.

Brian Parshall expressed the hope that the bound $p > 2h - 2$ can be weakened, but on Wednesday, Geordie Williamson explained a counterexample to the positivity statement for $G = GL(15)$ and $p = 2$, and expressed the hope that this counterexample will generalize to yield many more examples of failure of positivity. Both hopes cannot be correct; going forward, it will be interesting to see where they collide with each other!

Quasimap spaces and Frobenius kernels

A fundamental result in the geometric representation theory of small quantum groups is the work of Feigin, Finkelberg, Kuznetsov, Mirković, and others, which relates these representations to perverse sheaves on the space of quasimaps from $\mathbb{P}^1$ to the flag variety. Several workshop participants suggested that there should be a parallel theory in positive characteristic, relating to the representation theory of the first Frobenius kernel of a reductive

group. One working group, led by Galyna Dobrovolska and Martina Lanini, made significant progress during the workshop on pinning down the correct definitions and on formulating a precise conjecture.

Tilting modules and Quasi-hereditary Algebras

A working group involving Daniel Nakano, Brian Parshall, and Leonard Scott studied two different questions about representations of algebraic groups. They began by investigating what happens when tilting modules for a reductive group G are restricted to their associated finite Chevalley groups. Some progress was made in terms of looking at the general linear group. A basic question is whether one can parametrize the indecomposable summands that appear in the restrictions.

Later, this group investigated the category of rational B-modules where B is a Borel subgroup of G. This is a highest weight category using van der Kallen's excellent filtrations. The group attempted to formulate a Kazhdan–Lusztig theory to calculate Ext-groups between simple B-modules (which is an open question). It is anticipated that the Ext-group calculation will not be as dependent on the prime as in the reductive case.

Koszul duality

In a seminal 1996 paper, Beilinson–Ginzburg–Soergel established the celebrated *Koszul duality* relationship between (characteristic-0) perverse sheaves on the flag variety for G and those on the flag variety for the Langlands dual group. Two notable subsequent developments are the work of Bezrukavnikov–Yun, establishing a Koszul duality for affine flag varieties (still in characteristic 0), and that of Achar–Riche, establishing a Koszul duality in positive characteristic (still on finite-dimensional flag varieties). It is highly desirable to combine these—i.e., to establish a positive-characteristic, affine Koszul duality—as this would imply the Riche–Williamson conjecture on characters of tilting modules.

In this direction, during the Friday afternoon session, Shotaro Makisumi reported on his results on Koszul duality for Soergel bimodules and for sheaves on moment graphs. His methods work for arbitrary Coxeter groups and appear to subsume all the previous results of Beilinson–Ginzburg–Soergel, Bezrukavnikov–Yun, Riche–Soergel–Williamson, and Achar–Riche. It seems likely that Makisumi's work contains hints on how to attack the (still open) case of positive-characteristic, affine Koszul duality.

Highest-weight structure for perverse sheaves

Two working groups studied questions around the theme of highest-weight structure and quasihereditary algebras governing perverse sheaves on various spaces. One team, led by Peter McNamara, sought to build on the work of S. Kato, identifying algebraic notions such as "standard modules" for Ext-algebras with certain geometric constructions. Another team sought an intrinsic geometric explanation for the highest-weight structure appearing in various examples, including the nilpotent cone for GL_n, and the affine Grassmannian for any reductive group, and hypertoric varieties. By the end of the week, Anthony Henderson and Carl Mautner succeeded in obtaining a new proof of the highest-weight structure for the nilpotent cone.