Shape optimization with surface interactions

organized by
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Workshop Summary

Overview and focus

The workshop, sponsored by the AIM and the NSF, examined state-of-the-art techniques in spectral geometry with an emphasis on shape optimization problems. One of the goals was to attract the community to recent challenging optimization problems involving surface interactions, which are described by imposing appropriate boundary conditions.

The considered open problems can be logically separated into one large category with several sub-categories and two satellite categories.

- Eigenvalue optimization for
  - differential operators with surface interactions (Robin Laplacian and Dirac operator with infinite mass boundary conditions);
  - classical Dirichlet, Neumann, and Steklov problems for the Laplacian;
  - the Hermite operator;
  - buckling linear pencil and the bi-Laplacian.
- Optimal constants in geometric functional inequalities.
- Hot spots conjecture and related properties of eigenfunctions.

Morning talks, discussion sessions and group work roughly followed this classification.

There were 13 attendants from the US, 1 from South America, and 8 from Europe.

Morning talks

Each morning we had two introductory lectures. *Monday, June 17.*

[1.] **A. Henrot**: *Isoperimetric inequalities for eigenvalues with geometric constraints* In this talk, it was recalled how the Dirichlet, Neumann, Robin, and Steklov eigenvalue problems for the Laplacian are introduced. Classical results on shape optimization for these eigenvalue problems were surveyed. A long list of about 20 open problems was suggested. Some of these open problems has appeared very recently, while the others remain open for decades. This list is included into the collection of open problems posted on the website of AIM. **R. Benguria**: *GNS inequality for domains* This talk was devoted to a non-linear variational problem on a bounded domain $\Omega \subset \mathbb{R}^n$. This problem naturally leads to a Poincaré-Sobolev type inequality, best constant in which is still unknown. There is a motivation coming from physics to better estimate it for hypercubes. New bounds on this constant were presented. The non-linearity and the structure of the variational problem surprisingly yield that existence/non-existence of a minimizer depends on the smoothness and the shape of

Ω. In particular, a minimizer exists for any C^3-smooth domain, for elongated rectangles and for hypercubes in the space dimension $n \geq 10$. However, it does not exist for isosceles triangles. The remaining geometric settings are open. Based on joint works with C. Vallejos and H. Van Den Bosch.

Tuesday, June 18.

[1.]**C. Trombetti**: *Optimization of the Robin eigenvalue I* The aim of the talk was to discuss an eigenvalue optimization problem for the Robin Laplacian on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with the boundary parameter $\alpha \in \mathbb{R}$. Previously known results for $\alpha < 0$ were surveyed. It was conjectured by Antunes, Freitas, Krejčiřík that for $\alpha < 0$ and $n \geq 3$ the ball is a maximizer under the constraint on the area of the boundary of Ω. In this talk, a proof of this conjecture was presented under an additional constraint that Ω is convex. As a second topic, a Talenti-type result for $\alpha > 0$ was discussed. A striking observation is that such a result (if true) leads to an alternative proof of the Bossel-Daners theorem. **D. Bucur**: *Optimization of the Robin eigenvalue II* In this talk, first, the proof of the Bossel-Daners theorem for the Robin Laplacian with a positive boundary parameter was recalled and explained in detail. It was demonstrated how to modify it in order to get an isoperimetric inequality in a quantitative form with the Fraenkel asymmetry being involved. Second, the Saint-Venant inequality for the Laplace operator with Robin boundary conditions was discussed. Finally, isoperimetric inequalities for the first non-trivial Steklov eigenvalue were presented. It was shown how to extend the Weinstock inequality for higher dimensional convex domains.

Wednesday, June 19.

[1.] **K. Burdzy**: *Hot spots conjecture* The hot spots conjecture is attributed to J. Rauch and it states that the maximum of the first non-trivial Neumann eigenfunction is attained on the boundary of a domain. This conjecture is proved to be in general wrong, but it is now conjectured to be still true for simply-connected domains in two dimensions and for convex domains in higher dimensions. In this talk, first counterexamples in the class of non-simply-connected planar domains were demonstrated. Second, a probabilistic proof of the hot spots conjecture was presented for a rather general class of so-called Lip-domains in two dimensions. One of the aims of this talk was to demonstrate the strength of probabilistic methods in spectral theory. **R. Laugesen**: *Conformal mapping methods* In this talk, recent results on optimization for low Robin eigenvalues were presented. The key interest was to understand optimization of the second Robin eigenvalue. Under area/volume constraint, the ball is shown to be the optimizer for the boundary parameter lying in a specific interval. For some results to hold, the boundary parameter needs to be properly rescaled, which introduces an extra flexibility to the problem. Two techniques were discussed: the method of conformal maps à la Szegő in two dimensions and the method of Weinberger in two and higher dimensions. The proof by means of the conformal map was explained in detail. An additional aim of this talk was to make the community more familiar with complex-analytic methods in eigenvalue optimization. Based on joint works with P. Freitas.

Thursday, June 20.

[1.] **Z. Lu**: *Spectral geometry of quantum layers* The speaker delivered a survey on the state-of-the-art and recent progress in the analysis of curved quantum layers. Particular attention was paid to the existence of so-called geometrically induced bound states in tubular neighborhoods of hypersurfaces. A unified differential-geometric approach to the problem was presented. A number of striking geometric ideas were explained. These ideas are less familiar to and sometimes overlooked by the community working in spectral geometry for Euclidean domains. **S. Steinerberger**: *Zero sets of eigenfunctions Any linear combination $\sum_{k\geq n} a_k\phi_k$ of eigenfunctions $\{\phi_k\}_{k\geq 1}$ of a Sturm-Liouville operator has at least $(n-1)$-roots.* This result by Sturm was forgotten and then later rediscovered by Hurwitz. There is no direct analogue of this result for the Laplacian in higher dimensions expressed in terms of the number of nodal domains, as counterexamples show. The message of this talk is that the right quantity to be bounded from below is the Hausdorff measure of the zero set of a linear combination of eigenfunctions, a lower bound on which for the Neumann Laplacian was presented. The proof of this bound relies on some recent and interesting in its own right techniques in the transportation of mass theory. Possible improvements and perspectives were discussed. Based on a joint work with A. Sagiv.

Friday, June 21.

[1.] **T. Beck**: *Level sets of the first Dirichlet eigenfunction in convex domains* The central topic of the talk was the structure of the ground-state u_1 for the Dirichlet Laplacian on a planar convex domain Ω. Understanding the structure of eigenfunctions is always a very delicate problem. The interest of the speaker was to extract some information on the shape of the level sets for u_1. The most striking consequence of this analysis is that u_1 is not only log-concave, but also its restriction to a neighborhood of the maximum is concave in the usual sense. Some general methods for the analysis of the eigenfunctions were explained. **T. Ourmières-Bonafos**: *(Euclidean) Dirac operators and shape interactions* The first aim of this talk was to make the audience familiar with a spectral problem of growing interest for the Dirac operator, which comes from the analysis of graphene in two dimensions and the analysis of quark's confinement in three dimensions. A lot is already known about rigorous definition of the underlying self-adjoint operator and its asymptotic spectral analysis in various limits. Non-sharp bounds on the size of the spectral gap have also been recently obtained. The second aim of the talk was to draw the attention to some conjectured isoperimetric inequalities for the size of the spectral gap, which are supported by physical intuition and numerical tests.

Group work and afternoon activities

During the Monday afternoon, 23 specific open problems were suggested by participants in a session moderated by Burdzy. After a pre-selection made by the organizers on Tuesday afternoon, 8 of them were suggested for voting; the participants picked eventually 4 open problems and separated into groups. Discussions on these four problems (listed below) was led by Henrot, Krejčiřík, Ourmières-Bonafos, and Steinerberger, respectivelY. On Wednesday, the groups reported on progress and the group led by Henrot decided to dissolve and merge into the other three. The group led by Krejčiřík achieved some progress and the remaining analysis required only careful and cumbersome computations, which do not essentially require collaborative brainstorm. The group decided to dissolve as well and two new groups were created on Thursday. The problems selected for these two groups were among the pre-selected ones, but they have not gained enough votes in the first voting round.

Substantial progress was made by the group of Steinerberger towards improving the constant in the multi-dimensional Hermite-Hadamard inequality for convex and sub-harmonic functions. This was reported on Wednesday and the problem was further developed. Work groups continued on Wednesday with reports on Thursday. Two open problems from the initial preselected list were brought into discussion by Brandolini/Krejčiřík and by Mercedi Chasman on Thursday in order to split the attendants into four groups again.

On Friday we had final reports and after that the groups indicated the goals to be achieved after the end of the workshop.

Selected problems (full statements in the List of Open Problems):.

[1.]**D. Krejčiřík, V. Lotoreichik, Z. Lu**: *Bareket's conjecture and its analogue for triangles.* Given a bounded Euclidean domain Ω, let $\lambda_1(\Omega,\alpha)$ be the first eigenvalue of the Robin problem with a negative boundary parameter α. The long-standing conjecture that $\lambda_1(\Omega,\alpha)$ is maximized by the ball among all sets of equal measure has been recently disproved by Freitas and Krejčiřík by considering the eigenvalue asymptotics in balls and spherical shells as $\alpha \to -\infty$. It is still conjectured that in two dimensions, the disk maximizes $\lambda_1(\Omega,\alpha)$ among all simply connected Ω's with the same area. In three and higher dimensions, it is expected that the ball maximizes $\lambda_1(\Omega,\alpha)$ among all convex domains Ω with the same volume. A simplified discrete analogue of the above conjecture is that among all triangles of a given area the equilateral triangle maximizes the lowest Robin eigenvalue with the negative boundary parameter. **A. Henrot**: *Optimizing Neumann eigenvalues among convex sets of fixed perimeter.* Let $\mu_k(\Omega)$ be the k-th non-trivial Neumann eigenvalue on a domain $\Omega \subset \mathbb{R}^n$. Let $P(\Omega)$ be the area of the boundary of Ω. Prove the existence of optimizers for

$$\sup\{\mu_k(\Omega)\colon \Omega \subset \mathbb{R}^n \text{ convex},\ P(\Omega) = P_0\}$$

and

$$\inf\{\mu_k(\Omega)\colon \Omega \subset \mathbb{R}^n \text{ convex},\ P(\Omega) = P_0\}.$$

For $k = 1$, $n = 2$ and convex Ω's, it is conjectured that $P(\Omega)^2\mu_1(\Omega) \leq 16\pi^2$, where equality is achieved if Ω is either a square or an equilateral triangle. **T. Ourmières-Bonafos**: *Faber-Krahn for Dirac operator with infinite mass boundary condition.* The eigenvalue problem for the two-dimensional Dirac operator with infinite mass boundary condition is defined as

$$\begin{cases} -\mathsf{i}\partial_1 u_2 - \partial_2 u_2 = \nu u_1 & \text{in} \quad \Omega\,, \\ -\mathsf{i}\partial_1 u_1 + \partial_2 u_1 = \nu u_2 & \text{in} \quad \Omega\,, \\ \mathsf{i}\big(n_1 + \mathsf{i} n_2\big) u_1 = u_2 & \text{on} \quad \partial\Omega\,, \end{cases}$$

where $n = (n_1, n_2)^\top$ is the outer unit normal to a bounded smooth domain $\Omega \subset \mathbb{R}^2$. Prove or disprove that the smallest positive eigenvalue $\nu_1(\Omega)$ for the above eigenvalue problem is minimized by the disk among all domains of equal measure. **S.**

Steinerberger: *Hermite-Hadamard inequality in higher dimensions.* The classical Hermite-Hadamard inequality states that if f is a convex function on an interval (a,b) then

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(b)+f(a)}{2}.$$

Consider two generalizations of the above inequality to arbitrary dimensions. Let $\Omega \subset \mathbb{R}^n$ be a convex domain and let the function $f\colon \Omega \to [0,\infty)$ be either convex or sub-harmonic. In both cases there are dimensional constants $c_n > 0$ and $\widetilde{c}_n > 0$ such that

$$\frac{1}{|\Omega|}\int_\Omega f(x)\,\mathrm{d}x \leq \frac{c_n}{|\partial\Omega|}\int_{\partial\Omega} f(x)\,\mathrm{d}x \quad \text{and} \quad \int_\Omega f(x)\,\mathrm{d}x \leq \widetilde{c}_n|\Omega|^{1/n}\int_{\partial\Omega} f(x)\,\mathrm{d}x.$$

The second inequality follows from the first by an application of the isoperimetric inequality with $\widetilde{c}_n = \frac{c_n}{n}$. The problem is to find optimal constants c_n in the inequality. Steinerberger recently proved that the optimal constants for convex functions satisfy $1 < c_n \leq \frac{2}{\sqrt{\pi}}n^{n+1}$, $\frac{9}{8} \leq c_2 \leq 8$ and for subharmonic functions $\widetilde{c}_n \leq \frac{\sqrt{2}}{\pi}$. The first goal is to improve these bounds. There is a conjecture that the constant c_n for convex functions can be taken to be 1 if the center of mass of Ω coincides with that of $\partial\Omega$. It was suggested to disprove it. **D. Bucur**: *Faber-Krahn for buckling problem.* Consider the lowest eigenvalue of the buckling problem

$$\begin{cases} \Delta^2 u + \lambda u = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Prove that the lowest buckling eigenvalue $\lambda_1(\Omega)$ is minimized by the disk among all simply connected sets of equal area. The problem can be reformulated in terms of the eigenvalue problem

$$\begin{cases} \Delta u + \lambda_1(\Omega)(u - P_\Omega u) = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where P_Ω is an orthogonal projection in $L^2(\Omega)$ onto the subspace of harmonic functions. If the subspace is that of all harmonic functions one retains the eigenvalue of the buckling problem. Partial results exist if one instead projects only onto certain subspaces of harmonic functions. **B. Brandolini, D. Krejčiřík**: *Poincaré-Wirtinger extremal domain.* Given $\Omega \subset \mathbb{R}^n$ and a Gaussian weight γ, consider the Hermite operator $-\gamma^{-1}\nabla \cdot \gamma\nabla$ on the weighted space $L^2(\Omega,\gamma)$, subject to Neumann boundary conditions. It is known that the first non-trivial eigenvalue of this operator satisfies $\mu_1(\Omega,\gamma) \geq 1$ (this is the Poincaré-Wirtinger inequality). Prove that equality in the Poincaré-Wirtinger inequality is achieved only by infinite strips. With a priori assumption that Ω is contained in an infinite strip the result is known to be true.

After workshop update

When finishing this report, there appeared a preprint on a partial solution of the open problem suggested by S. Steinerberger:

([illegible]) Thomas Beck, Barbara Brandolini, Krzysztof Burdzy, Antoine Henrot, Jeffrey J. Langford, Simon Larson, Robert G. Smits, Stefan Steinerberger: *Improved Bounds for Hermite-Hadamard Inequalities in Higher Dimensions*, arXiv:1907.06122 [math.CA].