# Problems on holomorphic function spaces and complex dynamics

organized by
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Workshop Summary

This workshop was organized to host the first AWM Research Collaboration Conference for Women in Several Complex Variables (WinSCV, an AWM-ADVANCE Research Network). The overarching motivation for this was to strengthen the collaboration network of women in SCV – by providing them the opportunity to work on mathematical problems of common interest. The AIM format provided an excellent support structure for this effort. There were two expository talks every morning which introduced everybody — in an accessible manner — to the various mathematical topics of the workshop. During the first afternoon session, problems were solicited from the participants from broadly four themes: (1) Extension of the Hardy space theory to non-smooth domains, (2) Dimension of the Bergman space of a domain in $\mathbb{C}^n$, (3) Dynamics on cubic surfaces, and (4) 'Hedgehogs' for elliptic germs. Twenty three problems from these topics (and beyond) were contributed by the participants, out of which six were chosen to be studied at the workshop. During the rest of the afternoons, the participants broke up into small groups to work on these problems. These sessions gave all the participants an opportunity to collaborate and contribute in a friendly, energetic and trusting atmosphere. Since many of the groups plan to continue working together, the workshop was successful in meeting its goals of forging new collaborations and initiating new projects in SCV.

## *Elliptic germs in dimension* 2

Our project was exploring a result by Firsova, Lyubich, Radu and Tanase. On their recent paper 'Hedgehogs for neutral dissipative germs of holomorphic diffeomorphisms of $(\mathbb{C}^2, 0)$' they studied and proved the existence of 'hedgehogs' for germs in $(\mathbb{C}^2, 0)$ in which the derivative $F'(0)$ has two eigenvalues $\lambda, \mu$ such that $|\lambda| = 1$ and $|\mu| < 1$. The existence of 'hedgehogs' follows by considering approximations to $F$ by $F_k$, where $F_k$ is a perturbation of $F$ such that $F_k(0) = 0$ and the eigenvalues of $DF_k(0)$ are $\lambda_k, \mu$ such that $\lambda_k \to \lambda$ and $\lambda_k$ is a root of unity and $\mu$ as before. Our project is to prove a similar result for $F$ a germ in $(\mathbb{C}^2, 0)$ such that the derivative $F'(0)$ has two eigenvalues $\lambda, \mu$ such that $|\lambda| = 1$ and $\mu = 1$. As a replacement for the description of the dynamics of $F_k$ done by Ueda, we instead use the results of Hakim. We were able to find some counterexamples for a general result for *any* $F$, so we do need to impose some conditions on the higher order terms of $F$.

## *Critical locus and critical measures for Cantat-Loray cubic surfaces*

The group discussed the dynamics of a family of maps on cubic surfaces. This family of maps comes from a discrete Schrödinger operator. It is well-known and extensively studied by mathematical physicists. Serge Cantat studied the complexified version of the maps and using the tools of multidimensional holomorphic dynamics was able to derive new and important consequences for the dynamics of real maps. He also showed that this family of maps comes from representations of a Mapping class group of a sphere with 4 punctures and explored the relation to Panleve's VI equation. Cantat used similarities with Hénon maps to study the dynamics of the maps on cubic surfaces. Bedford and Smillie studied the dynamics of holomorphic Hénon maps and produced a sequence of 8 influential papers. In particular, they defined the critical loci in important dynamically defined regions and the critical measures. Firsova, Robertson and Lyubich obtained important results about the critical locus. Our group investigated the critical loci for Cantat-Loray maps. The group consisted of two tenure track members and two graduate students. As a result, we spent a substantial amount of time reviewing background material. We read and discussed Cantat's and Cantat-Loray's articles, as well as Bedford-Smillie article on the critical locus. We did calculations and filled in the details of the behavior of the maps at infinity, discussed algebraic stability and blow ups. We discussed the behavior of the Green's functions at infinity and discussed in details what is know about the critical loci of Hénon maps (results of Bedford, Smillie, Lyubich, Robertson and Firsova). Members of the group plan to continue to work on the project.

### *Dimension of weighted Bergman space for planar domains*

This group studied the dimension of weighted Bergman spaces for planar domains. It became clear quickly that, using Hörmander's $L^2$-methods, the weighted Bergman space of a planar domain equipped with a locally integrable weight function $e^{-\phi}$ was infinite dimensional whenever the domain admitted a bounded function $\psi$ such that $\psi + \phi$ was strictly subharmonic on the domain. The group also studied the problem for planar, bounded domains with weights which are not locally integrable. In fact, it was derived that in this case, the Bergman space is either trivial or infinite dimensional. In the later part of the week, the group turned towards the dimension problem of unweighted Bergman spaces for certain Hartogs domains. This problem has been studied in detail by P. Jucha. The group focused on examples of some remaining, open cases – and gained some understanding of these as they could be reduced to questions on the Bergman space dimensions for certain planar domains with weights.

### *Existence of bounded plurisubharmonic functions*

The existence of certain strictly plurisubharmonic, bounded functions on a domain in $\mathbb{C}^n$ is sufficient for the infinite dimensionality of the Bergman space of this domain. The groups intend was to study the questions whether one could construct bounded, plurisubharmonic functions whenever the Bergman space of the given domain is non-trivial. This problem led the group to a different problem: construct a domain for which the Bergman space does not separate points. No such domains exist in $\mathbb{C}$ unless the Bergman space was trivial to begin with. The group constructed an example in higher dimension: $\{(z,w) \in \mathbb{C}^2 : |w|^2 < (1+|z|^2)^{-1}\}$ for which the Bergman space does not separate points on the plane $\{w = 0\}$. This group dissolved after the first day.

### *Dimension of Hardy spaces on planar domains*

This group sought a potential-theoretic characterization of planar domains that admit a nontrivial Hardy space (the space of holomorphic functions $f$ such that $|f|^2$ admits a harmonic majorant), with the hope of establishing the following dichotomy: the Hardy space of a planar domain can either be trivial or infinite-dimensional. Based on a discussion of some known results, the group agreed to attempt two (competing) characterizations of such domains — one, the complement must be nonpolar, and other, the complement must be removable. To make progress on this problem, we needed to understand the growth conditions encoded in the definition of a Hardy space. We concluded that in order for $f$ to be in the Hardy space, a certain weighted $L^2$-norm of $f'$ must be finite, where the weight comes from the Green function of the domain. This observation became central to all the approaches subsequently discussed by the group. This also tied up to a different (but related) notion of Hardy spaces that has been studied more recently. In our final meeting at the workshop, we extracted three main questions from our discussions, and intend to continue working on this problem.

### *Natural data sets for Boundary Value Problems for holomorphic functions*

It is known that the holomorphic Hardy space is the natural (optimal) data space for the $L^p$-Dirichlet problem for holomorphic functions on a given domain $D \subset \mathbb{C}^n$, with $n \geq 1$. This group is seeking to characterize data spaces for other boundary conditions, such as Neumann; Robin, and more. During the workshop, a candidate was identified for the optimal data space of the Neumann boundary value problem for holomorphic functions, and work is in progress to prove the complete characterization.