Sarnak's conjecture

organized by
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Workshop Summary

The main subject of the workshop – Sarnak's conjecture on Möbius disjointness (of deterministic continuous observables) – by its nature, lies on a borderline between analytic number theory and ergodic theory. Recent years have witnessed impressive progress on Sarnak's conjecture (and, closely related to it, the Chowla conjecture) and the progress appears to go in tandem. The main aim of the workshop was to foster deep collaborations between people working on the ergodic theory side of the subject and those working on the purely arithmetic aspects. The workshop brought together top experts from both areas, in a proportion of approximately 1:1, and included a number of excellent young people working on questions tightly related to Sarnak's conjecture.

Every morning there were two talks (one talk on the number theory side and one on the ergodic theory side, each about 1,5 h long). An important feature (as required by the AIM workshop style) was to make the talks accessible to all participants but also to make familiar, on a possibly introductory level, the ergodic side with techniques of analytic number theory and vice versa. The talks were given by:

- Terence Tao and Nikos Frantzikinakis,
- Alexander Mangerel and Clemens Müllner,
- Joni Teravainen and Adam Kanigowski,
- Oleksiy Klurman and T. de la Rue,
- Adam Harper and Tamar Ziegler.

On Monday, the entire afternoon session was devoted to formulate open problems. Finally, there were gathered 24 problems and questions. Each of them was discussed on some introductory level and informally approved by all participants. On the ergodic theory side they can be grouped into questions about:

- validity of Sarnak's conjecture for a version with level sets of multiplicative functions and for a polynomial version,
- spectrum a Liouville system,
- possibility of the $\mathbb{T}^2$-system $(x, y) \mapsto (x, x + y)$ being a Liouville system,
- a Möbius version of the Jewett-Krieger theorem,
- a possible proof of uniform Möbius disjointness for horocycle flows,
- uncountability of preimages for the natural coordinatewise-squaring map between the subshift determined by the Möbiuos function and the subshift determined by its square,
- the existence of non-trivial Möbius systems for aperiodic multiplicative functions

In subsequent working groups (Tuesday-Friday), a significant progress has been made in understanding connections between Möbius disjointness and its version using level sets of

multiplicative functions. Some progress has been made in understanding relations between logarithmic and classical Möbius disjointness. Furthermore, discussions in working groups raised many other questions, including the validity of the Prime Number Theorem (PNT) for weakly mixing transformations or on the potential equivalence of Sarnak's conjecture with its Landauian version: $\sum_{n\geq 1}\frac{1}{n}\mu(n)f(T^n x)$ converges. The latter raised the problem of connections of this version of Sarnak's conjecture with the PNT in dynamics.

On the number theory side, there were many specific questions concerning various properties of the Liouville function, among them:

- Whether one can reduce H in the recent result of Matomaki-Radziwill-Tao on Fourier uniformity
- Whether the result of Matomaki-Radziwill-Tao on Fourier uniformity can be extended to nilsequences
- Whether arbitrarily long constant blocks appear within the sign pattern of the Liouville function
- Whether the set of primes p for which $\lambda(p-1)=-1$ is infinite
- Whether one can use the entropy decrement argument to obtain new bounds for the number of consecutive integers that are smooth.

Progress has been accomplished on the third and fifth question leading to a clarification of the main difficulty. Essential progress has been accomplished on the second question (yes an extension is possible) and some progress has been also made on the first question.

The workshop strengthened in a significant way the interaction between number theorists and ergodicians, as well as between various groups of researchers. We also hope that it was particularly beneficial for young participants: 60% of talks were given by brilliant very young researchers who also were very active in working groups. We expect that the progress achieved in working groups will lead to new collaborations and joint papers in the near future. In fact, including more classical interactions between ergodic theory (homogenous dynamics) and number theory, following the initiative of the organizers of the workshop, most of the participants were enthusiastic about applying for an MSRI semester to continue this collaboration at a longer distance.