Representation stability

organized by

Andrew Putman, Steven Sam, Andrew Snowden, and David Speyer

Workshop Summary

Summary. This workshop brought together researchers in topology, combinatorics, and representation theory to study topics related to representation stability. The notion of representation stability was introduced by Church–Farb in 2010 and has since acquired a large and quickly growing literature. What is more, the different communities that are interested in representation stability set up the foundations in different ways, rendering communication somewhat difficult. The vision for this workshop was to bring these different communities together to help clarify what is known and plot out future directions.

We were very successful at realizing these goals. The daily structure of the workshop comprised two broad survey talks in the morning and an afternoon session in which we broke into groups to work on open problems concerning representation stability. These problems came from a problem session we ran on Monday afternoon. We also held another problem session before lunch on Friday. An annotated problem list coming from these sessions is available on the conference website. Before the conference, the organizers also prepared a detailed annotated bibliography on the subject, which is also available on the conference website.

Talks. We had two broad survey talks each morning of the conference.

- We began on Monday morning with an introductory talk by Jordan Ellenberg that focused on some of the different applications and triumphs of representation stability, with a focus on the theory of **FI**-modules. We then had a talk by Andrew Snowden on twisted commutative algebras, which are an important alternate set of foundations for the subject (and include **FI**-modules as a special case).
- On Tuesday, Vic Reiner gave a talk on combinatorial aspects of the cohomology of pure configuration spaces of $\mathbb{R}^n$ that highlighted a number of open questions and conjectures. This was followed by a talk by Thorsten Heidersdorf on Deligne categories, which are yet another set of foundations for the theory of representation stability.
- On Wednesday, Jeremy Miller spoke on how homotopy theorists think about the cohomology of configuration spaces of manifolds, with a focus on the insights given by space-level localization. We then had a talk by David Speyer on the results about **FI**-modules that come from the classical theory of the representation theory of the symmetric group in characteristic 0.
- On Thursday, Nathan Harman gave a talk on the rapidly emerging picture of how much of representation stability works in finite characteristic. Next, Jennifer Wilson gave a survey talk on the cohomology of configuration spaces of manifolds (special cases of which had appeared in many previous talks).

- On Friday, Weiyan Chen gave a talk on applications of representation stability to arithmetic statistics. This was followed by a talk by John Wiltshire-Gordon on computational aspects of representation stability.

Algebraic questions. Nate Harman posed a question about the existence and uniqueness of integral structures on **FI**-modules. This question was quickly and completely resolved: an **FI**-module over the complex numbers need not admit a form over the rational numbers (and in particular, need not admit an integral form), but an **FI**-module defined over the rational numbers admits an integral form, and any two agree locally at all but finitely many primes.

A second algebraic question was also studied: is a **VI**-module of polynomial growth algebraic? This, along with an analogous question for $\mathbf{FS}^{\mathrm{op}}$-modules, was the subject of several days work for one of the groups. A number of basic insights were collected over the week that have clarified the picture greatly. For instance, we observed that in an $\mathbf{FS}^{\mathrm{op}}$-module of polynomial growth, the irreducible S_n representations appearing have bounded second row and polynomial multiplicities. The biggest partitions have stable multiplicities, and we suspect this is true for the smaller partitions as well, but that is still open. The main question is still very much open, and a number of participants have indicated interest in continuing to work on it.

Topological questions. The theory of twisted commutative algebras gives a vast generalization of **FI**-modules, but few examples have been found thus far. Jordan Ellenberg proposed the following example in the problem session: the homology of the pure configuration space of the cylinder $X = S^1 \times [0, 1]$ is known to be a finitely generated **FI**-module, where the action of **FI** roughly corresponds to bringing in a point from a boundary component (either one). Since there are two different boundary components, in fact, one gets an action of the category $\mathbf{FI}_2$, which is a 2-variable generalization of **FI**. Ellenberg pointed out that the first homology group is a direct sum of two projective **FI**-modules, and suggested that as an $\mathbf{FI}_2$-module, it is in fact a nontrivial extension of the two. This was proven by the group and a number of preliminary results about extensions of **FI**-modules as $\mathbf{FI}_2$-modules were obtained. They also considered other examples besides the cylinder, so it is likely there is a general phenomenon waiting to be properly understood.

Combinatorial questions. A number of participants worked on studying conjectures of Proudfoot, and of Hersh and Reiner, on combinatorial descriptions of the cohomology of ordered configurations of n distinct points in $\mathbb{R}^d$ (a space known as $\mathrm{PConf}^n\mathbb{R}^d$). One result which was established was a generalization of a result of Weiyan Chen, who had earlier shown by number-theoretic methods that the behavior of $H^i(\mathrm{PConf}^n\mathbb{R}^2)$ as a representation of S_n is "quasi-polynomial in i". The group was able to reprove this result by combinatorial means and generalize it to $\mathrm{PConf}^n\mathbb{R}^d$ for all d. This is a basic building block in studying configurations of points in more complicated manifolds. We tried several methods to prove analogous results for such configurations and found a collection of counter-examples which ruled out easy approaches. We also found several new restatements of Proudfoot's conjectures.

Deligne categories. Some of the participants worked on a specific problem bridging the Deligne categories for the symmetric groups $\mathrm{Rep}(S_t)$ with the category of algebraic representations of the infinite symmetric group $\mathrm{Rep}(S_\infty)$ studied in work of Sam and Snowden.

The main result they obtained is that a functor $\mathrm{Rep}(S_\infty) \to \mathrm{Rep}(S_t)$ whose existence is determined by certain universal properties is exact.