SEARCH EXPERIENCES IN LINEAR ALGEBRA AND NUMBER THEORY FOR UNDERGRADUATE FACUL

organized by
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Workshop Summary

At the Iowa Field of Dreams Conference for undergraduate faculty at HBCUs and Hispanic serving institutions, which took place last October at the University of Iowa, faculty participants formulated the concept of bringing together undergraduate faculty for a workshop to introduce them to current research opportunities in linear algebra and number theory that will equip them with the necessary tools to mentor students in undergraduate research. A proposal for such a workshop was submitted and approved by AIM. The workshop was held August 11-15, 2008, with 20 participants and 4 facilitators.

On the first day of the workshop, facilitators Bryan Shader and Leslie Hogben presented the linear algebra Lights Out game, and Estelle Basor and David Farmer gave talks on randomly generated matrices and the Riemann zeta function respectively. A training session on the free open-source mathematical software Sage was given by Michael Hansen on the second day of the workshop. Participants then split into two groups, one group working with Bryan and Leslie and the other group with Estelle and David for the remainder of the week. Both groups used Sage to explore different problems that were posed. Progress reports were given every day by each group.

The Projects

Linear Algebra Group – The Lights Out Game

The Lights Out game consists of a 5x5 grid of lights each operated by a button, where each light has two states, on or off. Pressing a button toggles the light and toggles its neighbors (diagonal neighbors are not affected). When the game starts, a random set of lights are switched on. Given this initial configuration of lights on and off, the aim of the game is to determine if it is possible to press a sequence of buttons in order to switch all the lights off, and if so, to determine the minimum number of steps required to obtain all lights out. The general project was to solve the problem using a system of linear equations. Other variations of the Lights Out game were investigated. One variation was to determine a solution for any given m by n grid of lights where pressing a button causes the rectilinear neighboring buttons to change states but not the state of the pressed button. Other variations of the game assume that each light has three states: black, grey or white, or that one has a triangular grid (or other types of graphs or trees) of light switches that is an m by n grid of switches on each level. For each variation of the problem, one is also interested in the rank of the solution set. These projects can also be defined in terms of graphs.

Number Theory Group – Random Matrices and the Riemann Zeta Function

Several projects were formulated to investigate various types of random matrices, the properties of their eigenvalues and their characteristic polynomials. One project focused on random Hermitian matrices. Sage was used to generate random Hermitian matrices and plot a histogram of their largest eigenvalues. Consider a random $n \times n$ Hermitian matrix, where the diagonal entries, x_{ii}, and the entries above the diagonal, $y_{ij} + iz_{ij}$, are taken from normal distributions, X, Y, and Z, respectively, where the variance of Y equals that of Z. A probability density function may be defined on the set of all $n \times n$ matrices over the complex field by $P(H) = c_n exp(-trace(H^2))$. The problem was to determine if c_n could be chosen so that the probability density function provides a good fit of the histogram of eigenvalues. Another project investigated the distribution of eigenvalues of random symmetric matrices. Wigner's semi-circle law states that for large $n \times n$ symmetric real matrices with elements taken from a distribution satisfying certain properties, the distribution of eigenvalues is the semicircle function. Sage was used to graph the eigenvalues in order to verify the semi-circle law. Variations of these and other projects on random matrices were investigated.

Several projects were also formulated to investigate the Riemann zeta function and its properties. One project was to understand the relationship between large gaps between zeros of the Riemann zeta function and large values of the zeta function on the critical line. A method used was to generate random unitary matrices, graph their eigenvalues in the complex plane, look for large gaps and see how large the characteristic polynomial gets between those eigenvalues. Then, see if there is a relation between the size of the gap and size of the characteristic polynomial in this gap. It is likely that these data behave similarly to the zeta function data. Another project investigated the most likely distribution of eigenvalues of the nn unitary matrices, given certain restrictions on the eigenvalues. There are two interesting cases, from the perspective of number theory. Given a large gap between two eigenvalues, where are the other eigenvalues expected to be? Given an eigenvalue of high multiplicity, what is the most likely configuration of the other eigenvalues? Additionally, several other projects also focused on the Riemann zeta function.

Discussion on Mentoring Students in Undergraduate Research

In addition to working on the research projects, workshop participants held discussions centered on the following questions:

What are the qualities of a good undergraduate research problem? How do you find and formulate good research problems for undergraduate students? What are the characteristics of a good mentor? What can Mathematical Institutes and Doctoral Universities do to support students and faculty in undergraduate institutions? What can undergraduate institutions do to prepare students for graduate school and for REUs?

Future Plans

Participants felt the need for more workshops of this nature. The facilitators are preparing a proposal for another one-week workshop for 25 participants, open to all who are interested. Possible topics for this workshop are graph theory and algebra. Sage training sessions are also being planned at the National Association of Mathematicians (NAM) MathFest, the Joint Mathematics Meetings and the Iowa Field of Dreams conference. Finally, workshop participants have formed collaborative teams and plan to continue the investigations started at the workshop.

Overall, the workshop was a success. The participants learned about current research problems in linear algebra and number theory and found several interesting problems to work on with undergraduate students upon returning to their institutions. Using Sage gave participants the opportunity to become familiar with a new free software package that they can continue to use in their research with students and possibly in some of their classes. The organizers would especially like to thank Leslie Hogben, Brian Conrey and David Farmer for all their efforts in putting the workshop together. We are also very appreciative of the many hours of teaching by Bryan Shader, Leslie Hogben, Estelle Basor, and David Farmer. Our special thanks go to Michael Hansen for providing valuable assistance in the use of the Sage software.

Reference:
Relevant workshop pictures, papers, and websites can be found here:
http://www.aimath.org/WWN/relant