Arithmetic reflection groups and crystallographic packings

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Workshop Summary

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This workshop was devoted to the complete classification of maximal hyperbolic arithmetic reflection groups and related questions. About 15 years ago, the number of such groups was shown to be finite, leaving open the possibility of fully classifying them in a way that makes them accessible for other applications, e.g., to the study of crystallographic sphere packings. Recent advances, both theoretical and algorithmic, made it plausible that with sufficient collaborative effort by experts in diverse areas from geometry/topology, dynamics, arithmetic groups, and number theory, this program could be completed. The goal of the workshop was to foster collaborations between people working in these different areas.

Every morning there were two talks. For the first three days, the talks broadly summarized the different approaches to this problem and some of the tools that are used. Talks on the last two days were more focused on particular problems and technical tools. The speakers were:

Mon: Alex Kontorovich and Alexander Kolpakov
Tue: Mikhail Belolipetsky and Ruth Kellerhalls
Wed: Nikolay Bogachev and Alice Mark
Thu: Anna Felikson and Mikhail Kapovich
Fri: Grant Lakeland and Benjamin Linowitz

On Monday afternoon, there was a moderated problem session, at which open problems were formulated and discussed. These problems were broadly organized into four categories, within which there were several sub-questions.

(1) *Vinberg's algorithm and decidability questions*
- Is reflectivity of an arithmetic lattice decidable?
- Can we bound the number of walls of the fundamental polytope of a reflection group in terms of a bound on its volume?
- For a non-reflective lattice, is there a lower bound on the number of walls we must find before we are guaranteed to have found an infinite order element?
- Is it possible to estimate the volume of the partial Dirichlet domain "in progress" during Vinberg's algorithm? (So that if this volume exceeds the known upper bound for volumes of reflective arithmetic lattices, the algorithm can halt.)

(2) *Classification*
- Classify all non-uniform maximal reflective lattices in a given fixed dimension n. Is it best to start with $n = 5$, the smallest dimension in which this has not yet been done, or $n = 19$, the largest dimension in which this has not yet been done?

- Same as the previous question, but restrict to lattices that are congruence.
- Suppose we wanted to create a repository of all known data for Coxeter polytopes in low dimension (like the LMFDB but for Coxeter polytopes). What information should it include and how should it be organized?
- Can we break the current record for highest dimensional cocompact reflective lattice, using something like $-(1+\sqrt{5}) \perp E_8 \oplus E_8$ or $-(1+\sqrt{2}) \perp D_n$?

(3) *Quasi-reflective and quasi-arithmetic groups*
- Can we classify quasi-reflective lattices in dimensions above 3?
- Do quasi-reflective lattices have any features that might suggest analgorithm for finding infinite order symmetries?
- Are there finitely many maximal quasi-arithmetic reflective lattices?
- Is there a combinatorial approach to groups generated by Cartan involutions in dimension above 4?

(4) *Apollonian-type groups*
- What are the various possible definitions generalizing to n-dimensions the classical Apollonian packing?
- Varying the values of the parameters λ and μ in the matrices generating the n-dimensional "super-Apollonian group" (as in Lagarias et al), when is the group discrete? A lattice? Quasi-arithmetic? When does it preserve a quadratic form? When does it have relations not already present in the Apollonian or dual Apollonian subgroups?
- Given a collection of matrices, can we determine its Zariski closure?

In the afternoons on Tuesday-Friday, four groups met each day that roughly corresponded to these four topics. On the first day, it was decided that reflectivity is, in fact, decidable. A lemma of Vinberg guarantees that the fundamental polygon of a non-reflective lattice has an infinite order symmetry, so the question then reduces to the question of whether Vinberg's algorithm always eventually produces enough faces that this symmetry is visible. If the algorithm continues to find corners, a symmetry will always appear, and even if it doesn't, including intersections outside hyperbolic space in the Minkowski model guarantees that a symmetry will be visible. That group spent the remainder of the week thinking about the second and third questions. It eventually found a way to refine Vinberg's algorithm using bounds related to volumes.

The group thinking about classification started with the non-uniform fixed dimension question, and decided to start in dimension 19. On Wednesday, a second group working under this broad topic worked on breaking the co-compact record. As a baby-example, they worked on $-(1+\sqrt{2}) \perp D_4$ instead. Progress was made in writing some code to solve this problem. Then the first group switched attention to the question suggested by N. Bogachev on classification of anisotropic reflective $\mathbb{Q}$-lattices of signature (3,1). This open problem appears to be the most accessible one. The group is planning to work on its solution in a near future.

The group working on quasi-arithmetic groups met on Monday and Tuesday. It was understood that there are infinitely many maximal quasi-arithmetic reflection groups in dimension 2, even the ones with bounded covolume. In dimension 3 the question is more subtle and the answer is unknown. The other questions were only briefly discussed, they are very interesting but appear to be hard.

The group working on Apollonian-type groups answered the question of when such a group preserves a quadratic form, and found bounds on the parameters that can be used to determine the signature of that form.