Zeros of random polynomials

organized by
Norman Levenberg, Doron Lubinsky, Igor Pritsker, and Maxim Yattselev

Workshop Summary

Report for the workshop "Zeros of Random Polynomials"

Talks were given by Ofer Zeitouni (Monday and Friday), Raphael Butez (Monday and Friday), Igor Pritsker, Pavel Bleher, Duncan Dauvergne, Igor Wigman, Turgay Bayraktar and Yen Do. A list of 15 problems was compiled on Monday afternoon, and we broke into 5 groups to work on selected problems.

D. Lubinsky, I. Pritsker, I. Wigman and X. Xie studied a range of problems on real zeros for random linear combinations of orthogonal polynomials on the real line. It is well known that Kac polynomials, spanned by monomials with i.i.d. Gaussian coefficients, have only $(2/\pi + o(1)) \log n$ expected real zeros in terms of the degree n. On the other hand, if the basis is given by orthonormal polynomials over a measure supported on the real line, then random linear combinations have $n/\sqrt{3}+o(n)$ expected real zeros. This phenomenon is related to the global distribution of zeros in this case, namely that the bulk of zeros is located near the support of the orthogonality measure on the real line. We discussed methods of finding sharper asymptotic results for the expected number of real zeros of random Legendre polynomials. It turns out this requires very precise asymptotics for Legendre polynomials and their derivatives up to the third order. These asymptotic results will enable us to obtain sufficiently sharp asymptotics for the associated reproducing kernels that appear in the Kac formula for the density of real zeros of random orthogonal polynomials. A sufficiently precise result on the expected number of real zeros of random Legendre polynomials will be also used to establish the first asymptotic result for the variance of real zeros in this case. We believe that the variance has the same order of growth as expectation, that is constant times the degree n. Proving the latter and finding that constant for the variance represent the second stage of the project. On the third stage, we would like to prove asymptotic normality for the distribution of real zeros of random Legendre polynomials. It is obvious that Legendre polynomials used as basis represent a convenient classical example, and further developments will deal with more general orthogonal polynomials. Finding the higher order correlation functions for real zeros, as well as all correlation functions for complex zeros, together with their asymptotics, represent additional interesting problems on zeros of random orthogonal polynomials. Moving beyond normal coefficients is yet another logical direction of research.

T. Bayraktar, T. Bloom, R. Butez, D. Dauvergne and N. Levenberg: Dauvergne (arXiv:1901.07614v1), building on joint work with Bloom, gave necessary and sufficient conditions for global convergence of the complex zeros of random orthogonal polynomials $G_n(z) = \sum_{j=0}^{n} \zeta_j p_j(z)$ where $\{\zeta_j\}$ is a sequence of i.i.d. non-degenerate complex random variables and $\{p_j\}$ are orthonormal polynomials in $L^2(\tau)$ where τ is a finite measure with non-polar compact support $K \subset \mathbb{C}$

which is regular on K in the sense of Stahl and Totik. Letting μ_{G_n} be the (normalized) zero measure of G_n, he showed that μ_{G_n} converges weakly to the potential-theoretic equilibrium measure μ_K

(1) almost surely if and only if $\mathbf{E}\log(1+|\zeta_0|)<\infty$ and
(2) in probability if and only if $\mathbf{P}(|\zeta_0|>e^n)=o(1/n)$.

(Indeed, he proved more: the same results hold if the sequence $\{p_j\}$ is *asymptotically minimal* on K which includes many classical families of polynomials; e.g., Chebyshev, Fekete, Faber, etc.). We considered the analogous univariate *weighted* situation. Given a compact set $K\subset\mathbb{C}$ and a continuous function Q on K such that the potential-theoretic weighted extremal function $V_{K,Q}$ is continuous, we take a finite measure τ on K satisfying a weighted Bernstein-Markov inequality on K: given $\epsilon>0$, there exists $C>0$ independent of n such that

$$||e^{-nQ}q_n||_K \le C(1+\epsilon)^n||e^{-nQ}q_n|_{L^2(\tau)}$$

for all polynomials of degree at most n, and for all n. Now the sequence $\{p_j\}$ becomes an *array* $\{p_{n,j}\}$ where $\{p_{n,0},...,p_{n,n}\}$ are orthonormal polynomials in $L^2(e^{-2nQ}\tau)$. Focusing on a convergence in probability result, the estimates of Dauvergne in the unweighted (sequence) case do not appear to suffice. However, a technique of Bloom to reduce or transfer a weighted problem in d variables to an unweighted, "homogeneous" problem in $d+1$ variables, appears to be helpful as an analogous (unweighted) convergence in probability result holds in $\mathbb{C}^d$ provided 2. is replaced by $\mathbf{P}(|\zeta_0|>e^n)=o(1/n^d)$ (section 7 of Bloom-Dauvergne arXiv:1801.10125v1). In a nutshell, given K,Q,τ as above, one considers the circled set in $\mathbb{C}^2$ defined as $Z:=\{(t,z)=(t,t\lambda)\in\mathbb{C}^2:\lambda\in K,\ |t|=e^{-Q(\lambda)}\}$. For $q_n(z)$ a univariate polynomial of degree n, $Q_n(t,z):=t^nq_n(z/t)$ is a homogeneous polynomial of degree n in $\mathbb{C}^2$; the measure τ lifts to an unweighted, bivariate Bernstein-Markov measure $\tilde{\tau}$ on Z; and there is a relationship between the weighted univariate extremal function $V_{K,Q}$ and the bivariate, unweighted extremal function V_Z.

H. Aljubran and M. Yattselev: Yattselev and Yeager (arxiv:1711.07852) have studied random polynomials of the form $P_n(z):=\sum_{i=0}^n\eta_i\varphi_i(z)$, where η_i's are i.i.d. real Gaussian random variables and $\{\varphi_i\}$ are orthonormal polynomials on the unit circle. A sufficient condition on the decay of the recurrence coefficients of $\{\varphi_i\}$ was obtained to guarantee that the expected number of real zeros of these polynomials is equal to $\frac{2+o(1)}{\pi}\log n$ (as in the Kac case). The expected behavior of the complex zeros was studied as well. Further, Aljubran and Yattselev (arxiv:1809.04948) have obtained the full asymptotic expansion for the expected number of real zeros of $P_n(z)$ when the recurrence coefficients of $\{\varphi_i\}$ decay exponentially (again, similarly to the Kac case derived by Wilkins). During the workshop we worked on extending these results to the case where $\{\varphi_i\}$ are Geronimus polynomials (all the recurrence coefficients are equal to a non-zero constant) and to the case where $\{\varphi_i\}$ are replaced by $\{\varphi_i'\}$.

Y. Do, H. Nguyen and O. Nguyen: Our group was interested in establishing some asymptotical estimates for the variance of the number of real roots of random algebraic polynomials with independent coefficients. More specifically, we would like to understand if there are general conditions that would guarantee that the variance is asymptotically comparable to the expected number of real roots. This question turns out to be quite broad so we decided to focus on well known models for which the answer was not known, and made some initial progress. In particular, we developed strategies to study these questions for generalized Kac polynomials and the Weyl polynomial.

P. Bleher, N. Cook, H. Nguyen and O. Zeitouni: Let $P_n(z) = \sum_{i=0}^n a_i z^i$ where a_i are zero mean iids, $z \in \mathbb{C}$. Konyagin and Schlag showed that if a_i are bernoullis then $M_n = n^{1/2} \min_{|z|=1} |P_n(z)|$ is of order 1. The problem we looked at is whether one can hope to say something about the distribution of M_n. We looked at the case of Gaussian coefficients, where better tools are available. We sketched a proof, based on two moments approximation to Poisson, for the fact that the number of points z of the form $e^{2\pi ikK/n^2}$, $k = 1, \ldots, n^2/K$ for which the value of $|p_n(z)|$ is smaller than $\epsilon/n^{1/2}$ is Poisson of parameter ϵ/K, and similar results hold for other strips. Then, we developed the idea that each point in the point process generates a close-by minimum by linear interpolation (with independent slopes) of the real and imaginary parts, and the fact that the function is roughly linear, locally. All in all, we hope these ingredients to yield a limit law. Work on this will continue.