Topological Complexity of Random Sets

organized by
Robert Adler, Jonathan Taylor, Shmuel Weinberger, and Steve Zelditch

Workshop Summary

This workshop collected researchers from a wide variety of areas in the mathematical sciences, including algebraic topology, probability, analysis, geometry and stochastic processes, theoretical and mathematical physics, and statistics. What united the participants was a common interest in a variety of topics at the boundary where topology and probability meet, and a desire to develop lines of attack that would prove fruitful in studying them.

It was clear from the early planning days of this workshop that it was, in some sense, a high risk enterprise. In the free format of AIM meetings, built on morning lectures and afternoons devoted to small working groups, there was always the danger that participants from the different disciplines would spend the afternoons in small, homogeneous groups rather than trying to use them to cross boundaries. The fact that this did not happen was in a large part a credit to the way the AIM staff forced the organisers to structure the afternoon sessions, and, even moreso, to the intellectual curiousity of the participants and their desire to exploit this rather unusual opportunity to expand their own interests. The final result was a synergy between the participants and their areas that more than fulfilled the original hopes of the organisers.

As one might expect, it took a while to develop this synergy. The six lectures of the first three mornings covered the six quite different areas of random fields and topology, algebraic topology and persistent homology, the Gaussian kinematic formula, random metrics, nodal domains of random spherical harmonics, and topological data analysis. The first two afternoons, which in the standard AIM structure, are when working groups begin to form, were actually closer to the traditional lecture style, as participants tried to develop common language and absorb one another's mathematical culture. It was really only on Wednesday afternoon that true interdisciplinary work began, and it is probably instructive to describe two of the main scenarios in which this happenned.

In one of the working groups, operating under the loose title of "Random fields, critical points and persistence" two of the youngest participants of the meeting, PhD students Omer Bobrowski (Technion) and Strom Borman (Chicago) presented a formula that they had derived the previous evening. This formula gave an explicit expression for the mean value of what is best described as a signed, weighted, Euler characteristic of the bar codes (measures of the persistent homology) of the excursion sets of real valued Gaussian random fields above a general level.

After seeing the result, the response of the more senior 'experts' in the audience whether topologists or probabilists, was unanimous. The result *had* to be wrong. Each argued from his own area of expertise, and over an hour was invested in checking the proof and searching for the 'mistake'. However, Bobrowski and Borman bravely held their ground, refusing to accept proof by intimidation, until, by the end of the day, all were convinced that the result had to be true, but no-one really understood it. By Friday afternoon, however, after much animated discussion, the topological significance of the result became clearer, statistical applications of it were being discussed, and it was starting to become clear that Wednesday's formula was a special case of a more general structure. It was also clear by then that the formula had precursors, appearing as special cases, in areas such as the theory of charged particles in Coulomb gases and in the study of large random matrices. This frenetic activity was a direct consequence of the broad spread of interests of the participants, and Bobrowski and Borman are currently working on extending their formula and building a theory around it. It should be emphasised that their result represents the first of its kind: a precise, simple, closed form expression for the mean value of a quantitative descriptor of the persistent homology of Gaussian excursion sets.

At the same time as the Bobrowski-Borman result was being hotly debated in one group, an equally animated discussion was going on in another group, meeting under the heading of "Random waves". Here, theoretical physicists Mark Dennis and Alex Barnett were describing certain filamentary structures observed in simulations of superpositions of random waves and of random spherical harmonics. Their question was whether or not the more mathematically inclined participants could turn these observations into theorems, primarily by identifying formal topological characterisations of what the eye saw and then studying these.(It rapidly became clear that what the eye sees is actually misleading. For example, while the eye sees long, connected filaments of mass above threshold levels, it turns out that the filaments are actually formed by long chains of small, isolated excursion sets.)

Over the following two days, this question became one of the most heated of the workshop. Ultimately, it was concluded that this phenomenon was related to the singular support of the spectral measure of the process, as late night simulations by Dennis and Barnett showed that other processes whose spectral measure lives on a curve showed similar filamentary structure. This is likely related to the observation that when summing over eigenfunctions with eigenvalues less than some λ no filamentary structures appear, but when summing over eigenfunctions with eigenvalues exactly λ filamentary structures appear. Opinions and arguments (often of the same individual) ranged over the spectrum between believing that theorems formalising these observations could be stated and proven, to claiming that the visual phenomena were artifacts of the simulation. By the end of the meeting, despite many blackboard arguments by mathematicians such as Misha Sodin and Jonathan Taylor, and more late night generation of impressive pictures by Dennis and Barnett, no final conclusions were reached. It was noted that the directions of filaments were highly correlated with the peaks of the "white noise" on the circle used in simulating the random plane waves. This led to the suggestion that the filaments may be related to similar structures noted in the literature when simulating Brownian sheet, although this would not necessarily explain the filamentary structure seen in random eigenfunctions where there is no analogy of a spectral representation theorem.

The tools suggested for formal results included Radon projection, searching for maximal-length lines in tubes around excursion sets, and scaling of the excursion sets simultaneously in wavelength and value. However, more work needs to be done, and will be done, to establish an appropriate theory. Once again, the success of this activity was a direct result of the inhomogeneous backgrounds of the participants.

In addition to the above two topics, which seemed to generate the most – or at least the noisiest – interest, there was considerable progress on a number of other topics as well.

In his Monday morning lecture, Shmuel Weinberger began with a broad introduction to persistent homology, showing its connections to other areas of mathematics, from geometric group theory to diffential geometry to algebraic topology as well as discussing ways that these topics might and benefit from probabilistic methods., During the talk, combining geometric arguments with energy estimates and a basic theorem of Cohen-Steiner, Edelsbrunner, and Harer, he defined a new homotopy invariant of finite complexes. A theorem of Gromov about closed geodesics on Riemannian manifolds whose fundamental group has unsolvable word problem shows its nontriviality, while classical results of Morse and of Bott lead to calculations for spheres and simply connected Lie groups.

In a related development, immediately following the workshop Weinberger succeeded in giving calculations for the H_0 and H_1 barcodes of the loopspace of a finite simply connected complex. Both David Allen and Peter Bubenik have since expressed optimism in using spectral sequence tools and simplicial model techniques for this problem. As there are some interesting examples and important potential applications in place, it is planned that these invariants will be further investigated in the future.

In a Wednesday morning talk, Misha Sodin explained his recent work with Nazarov on a number of nodal domains of random spherical harmonics and tried to emphasize the phenomenon of uniform lower semicontinuity of topological characteristics of nodal sets of random functions of several variables outside a set of functions of very small probability. This work seems to be the first rigorous work in that area which is full of intriguing (and probably very difficult) open questions. The same afternoon, Baryshnikov, Sodin, and Yomdin had a very useful and extensive discussion on some of these questions.

In a Thursday morning talk which, ab initio, mixed topology and probability, Matthew Kahle discussed the configuration space of n hard discs of radius r placed at random in a box, a model which is important in statistical physics and to the Metropolis algorithm in statistics. He raised the question as to whether or not this configuration space can be homotopy equivalent to the classical configuration space of n distinct labeled points in the plane, a very well understood space in algebraic topology. Later that day, Kahle, Yuliy Baryshnikov and Peter Bubenik were able to establish this equivalence when $r = O(1/n)$. They also made some headway into the question of bounding, from above and below, the topological complexity of these spaces, in the sense of Lusternik-Schnirrelmann category, which is the minimum number of contractible open sets needed to cover a space, as well as in deriving n and r dependent estimates of Betti numbers. As a result of this activity, an AIM SQuaREs proposal is in the offing, between participants from related but different disciplines who had never previously thought of working together.

Yet another follow up from a morning talk, this time by Steve Zelditch, occured in the area of random (Riemannian, Kahler and Bergman) metrics. In his talk, Zelditch presented a new definition of 'random Bergman metrics of level k' in a fixed conformal class on a surface by pulling back the Fubini-Study metric under random Kodaira embeddings. Bergman metrics give finite dimensional approximations to all smooth metrics, much as polynomials give approximations to smooth functions. The Bergman metrics of level k are in one-one correspondence with postive Hermitian matrices of rank $k + 1$. One may define probability measures on the space of smooth metrics by taking limits of sequences μ_k of probability measures on the Bergman metric spaces. Following the talk a number of formal and informal discussion groups formed and a number of interesting developments occured.

In particular, the relationship of the random Bergman metrics to those that arise in Liouville quantum gravity considered in the recent work of Duplantier-Sheffield and Hu-Miller-Peres became clearer. More specifically, Zelditch and Sam Klevtsov calculated correlation functions for some simple sequences μ_k arising in random matrix theory, such as Wishart measures. These correlation functions will never be singular enough to match the physicists' definitions of random metrics. Hence one needs to use geometric weights defined by approximations to energy functionals.

Still within the framework of random metrics, Dmitry Jakobson described some of his recent joint work on scalar curvature and Q-curvature for random Riemannian metrics in a fixed conformal class on a compact Riemannian manifold, assuming suitable Sobolev regularity. In consequent group discussion it was discovered that certain rare event probabilities that are needed for this theory can be estimated very well using existing results of Adler-Taylor for random fields on homogeneous manifolds. Furthermore, it became clear that extending some of the Adler-Taylor results would lead to similar estimates in the hitherto much harder case of non-homogeneous manifolds. By the end of the week it seemed certain that this blend of different approaches would continue into the future, leading to new directions, such as the possibility of studying sectional and Ricci curvatures of random metrics on high-dimensional manifolds.

Another topic that arose a number of times during the workshop was the Gaussian kinemetic formula, originally established in Taylor's 2001 McGill thesis, that both Adler and Taylor discussed in depth in their talks on Monday and Tuesday mornings. It appeared, as just noted, in the random metrics discussions, and it, and a integral geometric result of Baryshnikov, were what was behind the original formulation and proof of the Bobrowski-Borman formula described above. During the workshop it became clear that it would be useful to have a version of this result for random holomorphic functions, and, following detailed discussions with Taylor, Zelditch and Shiffman plan to work on this.

An final example of a (quite unpredicted) project which began during the workshop involved an AIM staff member, Estelle Basor, and a participant, Steve Evans. They began a project of looking at the indices of Toeplitz operators on the Hardy function space H^2 of holomorphic functions. The index of a Toeplitz operator is a topological object, but it can also be calculated as the winding number of an associated function. Taking this function to be a pinned, complex Brownian motion leads to the study of random Toeplitz operators. Much is known about the asymptotics of the winding numbers of complex Brownian motion due to the work of Spitzer, Pitman and Yor, and the plan is to exploit these results to obtain and study some interesting families of random Toeplitz operators.

All of the above examples should help paint a picture of a very successful workshop, that got people from very different areas of the mathematical sciences working together on problems that none of them would have been likely to tackle alone. In that sense, the workshop was a resounding success. Consequently, plans are already afoot to follow up on this workshop by creating an organised network of researchers interested in further pursuing and developing the probability/topology interface. But that is another story, the details of which we hope to be able to tell soon.