Analysis and geometry on pseudohermitian manifolds

organized by
Sorin Dragomir, Howard Jacobowitz, and Paul Yang

Workshop Summary

patterns intersections,decorations.markings

use path for main/.code=, use path for actions/.code=, use path/.style=use path for main=1, use path for actions=1,

The workshop activities consisted of morning lectures, a selection of open problems proposed by the participants, and afternoon working sessions in which the proposed open problems were discussed. The afternoon discussions led to a better understanding of the proposed problems and paved the road towards possible solutions.

The morning lectures covered Cauchy-Riemann and pseudohermitian geometry, complex analysis of functions of several complex variables, and subelliptic partial differential equations.

The Morning Lectures

- *The Q-prime curvature in CR geometry* by **Jeffrey Case**

A survey of the key properties of the Q-prime curvature for pseudo-Einstein manifolds. This invariant possesses many formal analogues with the critical Q-curvature in conformal geometry. For example, its integral is a nontrivial global CR invariant which is closely related to the Burns-Epstein invariant and the problem of prescribing Q-prime to be constant is closely related to Moser-Trudinger inequalities on CR pluriharmonic functions.

- *Borderline Sobolev inequalities after Bourgain-Brezis and applications* by **Sagun Chanillo**

The borderline Sobolev inequalities obtained by Bourgain-Brezis are extended to Riemannian symmetric spaces and to nilpotent groups. Applicatiions include Strichartz type inequalities for wave and Schrodinger systems, the $2D$ incompressible Navier-Stokes equations, and Maxwell's equations of electromagnetism. Several open problems like the sharp constants and optimizers for the inequalities, the inequalities in other settings like the pseudohermitian case, the case of Riemannian manifolds etc. were also discussed.

- *Hypersurfaces in the Heisenberg group* by **Jih-hsin Cheng**

An analysis of singular sets and characteristic curves in the three dimensional Heisenberg group with proofs of a Bernstein-type theorem, nonexistence of hyperbolic surfaces with bounded p-mean curvature, and the Pansu conjecture. For higher dimensions the notion of umbilicity was introduced and it was shown that Pansu spheres are the only umbilic spheres with positive constant p-mean curvature up to Heisenberg translations.

- *Regularity at the boundary for subelliptic PDEs* by **Giovanna Citti**

A review of classical results in the subriemannian setting and a discussion of recent work on Schauder estimates at the boundary in Carnot groups.

- *Pseudohermtian geometry* by **Sorin Dragomir**

A review of the main tools in pseudohermitian geometry: the Tanaka-Webster connection, Fefferman's metric, and the Graham-Lee connection. A discussion of the Lee Conjecture that compact strictly pseudoconvex CR manifolds whose CR structure has a vanishing first Chern class should admit a pseudo-Einstein contact form.

- *Aspects of spectral theory of complex Laplacians* by **Siqi Fu**

A review of spectral theory of complex Laplacians, in particular, of the $\bar{\partial}$-Neumann Laplacian and the $\bar{\partial}_b$-Laplacian with emphasis on the interplay between spectral behavior and geometric properties of the underlying complex or CR manifolds. Regularity and spectral properties of the $\bar{\partial}$-Neumann Laplacian and the $\bar{\partial}_b$-Laplacian have been shown to be more sensitive to the underlying geometry than the usual Dirichlet or Neumann Laplacian. The stability of the Bergman kernel on towers of coverings of complex manifolds was presented as an application

- *Around the Sobolev inequality* by **Andrea Malchiodi**

A discussion of some functional inequalities of critical type, corresponding to the borderline cases of Sobolev type embeddings for Dirichlet p-energies. In the Euclidean case extremals are classified, while for the Heisenberg case this is known only for $p = 2$. For pseudohermitian manifolds, the question of attainment of the conformal Sobolev-Webster quotient is related to the embeddability of the structure.

- *The Kepler problem on the Heisenberg group* by **Richard Montgomery**

Newton posed and solved "the Kepler problem", thereby deriving Kepler's three laws from a differential equation. Lobachevsky did the same in hyperbolic space. Kepler's problem on the Heisenberg group can be expressed as a Hamiltonian ODE. Due to the homogeneity of the potential any periodic solution must have total energy zero. Here the system is completely integrable and existence proofs of periodic solutions admitting k-fold cyclic symmetry groups follow.

- *Open problems (and some recent results) in analysis and geometry of worm domains* by **Marco Peloso**

The fundamental facts about analysis and geometry on the Diederich-Fornaess worm domain. This domain is a smooth, bounded pseudoconvex domain in $\mathbb{C}^2$ that presents a number of peculiar pathologies and in many cases it is the sole counterexample in function theory of several variables. Recent progress in analytical and geometrical properties of such domain and its model versions was presented.

• *Survey of interior regularity results for subelliptic PDEs* by **Luca Capogna**

A broad overview of interior regularity results for second order linear and nonlinear scalar PDEs of subelliptic type, starting from L. Hörmander's landmark hypoellipticity result, and following the development of the topic through the work of E.M. Stein and collaborators. Emphasis was given to the relation between regularity of solutions and metric geometry aspects of the underlying Carnot-Carathéodory spaces.

The Collaborative Groups

Report on progress obtained so far on the problem: Existence, regularity and geometric properties of exponentially subelliptic harmonic maps

Elisabetta Barletta and **Sorin Dragomir**. Exponentially subelliptic harmonic (e.s.h.) maps are C^∞ maps from a strictly pseudoconvex CR manifold $(M, T_{1,0}(M))$ endowed with a positively oriented contact form θ (i.e. the Levi form G_θ is positive definite) into a Riemannian manifold N, with the Riemanniann metric h, which are critical points of the functional

$$E_\Omega(\phi) = \int_\Omega \exp\left[\sigma_b(\phi)\right]\, \theta \wedge (d\theta)^n \,,$$

$$\sigma_b(\phi) = \frac{1}{2}\,\mathrm{trace}_{G_\theta}\left(\Pi_H \phi^* h\right), \quad \Omega \subset\subset M, \quad \phi \in C^\infty(M, N).$$

The following results have been proved while attending the A.I.M. workshop:

i) A geometric interpretation of e.s.h. maps: the vertical lift $\Phi = \phi \circ \pi : C(M) \to N$ [to the total space of the canonical circle bundle $S^1 \to C(M) \xrightarrow{\pi} M$] of any e.s.h. map $\phi : M \to N$ is exponentially harmonic (e.h.) as a map of the Fefferman space $(C(M), F_\theta)$ into the Riemannian manifold (N, h) and conversely, the base map $\phi : M \to N$ associated to any S^1 invariant e.h. map $\Phi : C(M) \to N$ is an e.s.h. map of (M, θ) into (N, h).

ii) The Euler-Lagrange equations of the variational principle

$$\delta \int_\Omega \exp\left[e_b(\phi)\right]\theta \wedge (d\theta)^n = 0$$

may be locally written

$$-\Delta_b \phi^\alpha + \sum_{a=1}^{2n} E_a(\phi)\, E_a(\phi)\Big(\left\{ \begin{array}{c} \alpha \\ \beta\gamma \end{array} \right\} \circ \phi\Big) + G_\theta\big(\nabla^H e_b(\phi),\, \nabla^H \phi^\alpha\big) = 0.$$

Here $\{E_a : 1 \le a \le 2n\}$ is a local frame of the Levi, or maximally complex, distribution $H(M)$ of M and turns out to be a Hörmander system of vector fields, while Δ_b is the *sublaplacian*, a subelliptic operator of order $\epsilon = 1/2$ coinciding locally with the Hörmander operator (Hörmander's "sum of squares") associated to the Hörmander system $\{E_a : 1 \le a \le 2n\}$. As such the study of existence and regularity properties of weak solutions to the e.s.h. map system revealed as culturally appropriate for a workshop devoted to the interaction of pseudohermitian geometry and subelliptic theory.

iii) When $N = S^m$, the standard sphere in $\mathbb{R}^{m+1}$, the e.s.h. map system was re-written as

$$-\left(\nabla^H\right)^*\left(\exp\left[e_b(\phi)\right]\nabla^H\phi^j\right) + G_\theta\left(\nabla^H e_b(\phi),\, \nabla^H\phi^j\right) = 0, \tag{0.1}$$

a form suitable for introducing a notion of *weak solution* to the e.s.h. map system. Precisely let $\Omega \subset M$ be a domain such that $\mathrm{Vol}(\Omega) = \int_\Omega \theta \wedge (d\theta)^n < \infty$. Then $\phi : \Omega \to N$ is a *weak e.s.h. map* if $\phi^j \in W_H^{1,2}(\Omega) = \mathcal{D}\left(\nabla^H\right) \subset L^2(\Omega)$ and

$$\left(\exp\left[e_b(\phi)\right]\nabla^H\phi^j\,,\,\nabla^H\varphi\right)_{L^2} = 2\int_\Omega e_b(\phi)\,\exp\left[e_b(\phi)\right]\phi^j\,\varphi\,\theta\wedge(d\theta)^n\,,$$

$$\forall\,\varphi \in C^\infty(\Omega).$$

iv) *F. Hélein's trick* still works for maps $\phi : \Omega \to S^m$ with values in spheres, for the e.s.h. map system may be recast as

$$\left(\nabla^H\right)^*\left(\exp\left[e_b(\phi)\right]\nabla^H\phi^j\right) = \sum_{j=1}^{m+1}\left(\nabla^H\right)^*\left(\phi_j\,E_{i,j}\right), \tag{0.2}$$

$$\left(\nabla^H\right)^* E_{i,j} = 0,$$

i.e. the second term in (0.1) may be written as a *generalized divergence.* The fields $E_{i,j}$ that do the job are

$$E_{i,j} \equiv \phi_j V_i - \phi_i V_j\,, \quad V_i = \exp\left[e_b(\phi)\right]\nabla^H\phi_i\,.$$

The form (0.2) of the e.s.h. map system may allow for duality lemmas and to successive Caccioppoli type estimates enabling one to conjecture that each weak e.s.h. map

$$\phi \in W_H^{1,\infty}(\Omega, S^m) = \bigcap_{p\geq 1} W_H^{1,p}(\Omega, S^m)$$

is locally Hölder continuous.

Among the participants **Sagun Chanillo**, **Giovanna Citti** and **Luca Capogna** showed interest to the problem, an interest which may lead to collaboration in the near future. To exhibit the relationship to the third area in workshop's theme (that is complex analysis of functions of several complex variables) one intends to study the boundary behavior of e.h. maps $\Phi : \Omega \to N$ from a smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ into a Riemannian manifold. One conjectures that at least for particular classes of e.h. maps (say with vanishing normal derivatives) boundary values are e.s.h. maps $\phi : \partial\Omega \to N$.

Report on the progress obtained so far on the problem Semipositivity of the Paneitz operator

It follows from the work of Kohn, Boutet de Monvel, and Burns that every compact strongly pseudoconvex CR manifold M^{2n-1} of dimension $2n-1$ is CR embeddable into some complex Euclidean space $\mathbb{C}^N$ when $n \geq 3$. The case when $n = 2$ is more subtle: M is embeddable if and only if the tangential Cauchy- Riemann operator $\bar\partial_b\colon L^2(M) \to L^2_{0,1}(M)$ has closed range, which in turn is equivalent to the fact that the Kohn Laplacian $\Box_b = \bar\partial_b^*\bar\partial_b\colon L^2(M) \to L^2(M)$ is gap positive, that is, the spectrum of $\Box_b$ lies outside of $(0,\varepsilon)$ for some $\varepsilon > 0$. It is thus important to find sufficient and necessary conditions for the closed range property of the $\bar\partial_b$-operator. Some necessary and sufficient conditions were given in terms of the CR Paneitz operator in [ChChY, CChY]. The Paneitz operator appears naturally in conformal geometry. Its CR analogue also arises naturally in the study of pluriharmonic functions and the degenerate Laplace-Beltrami operator on strongly pseudoconvex domains [L, GL].

On a 3-dimensional strongly pseudoconvex CR manifold with a pseudo-hermitian structure, the Paneitz operator is given by

$$P = \frac{1}{4}(\Box_b \overline{\Box}_b - 2Q) = \frac{1}{8}(\Delta_b^2 + T^2 - 4\,\mathrm{Re}\,Q)$$

where $\Delta_b = \nabla_b^2 = \Box_b - iT$ is the sub-Laplacian, T the Reed vector field. The operator Q is the purely holomorphic second-order operator defined by

$$Qu = 2i(A^{11}u_1)_{,1},$$

where A^{11} is the Webster torsion. It is shown in [ChChY] that on a strongly pseudoconvex CR manifold M^3 with semi-positive Paneitz operator and strictly positive Webster curvature, the non-zero eigenvalues of the Kohn Laplacian $\Box_b$ is bounded from below by the minimun of the Webster curvature. Thus, $\overline{\partial}_b$ has closed range and as a consequence M is embeddable.

The Rossi sphere is the unit sphere $S^3 = \{(z_1, z_2) \in \mathbb{C}^2;\ |z_1|^2 + |z_2|^2 = 1\}$ in $\mathbb{C}^2$ endowed with the CR structure given by

$$L_t = L + t\overline{L} \tag{0.3}$$

where t is a constant such that $0 < |t| < 1$ and $L = \bar{z}_1 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial z_1}$ is the standard CR structure on S^3. It is an archetype of a non-embeddable 3-dimensional strongly pseudoconvex CR manifold. Let (S^3, L_f) be the unit sphere S^3 endowed with the CR structure

$$L_f = L + f(z_1, z_2)\overline{L} \tag{0.4}$$

where $f(z_1, z_2)$ is a smooth function on S^3 such that $|f(z_1, z_2)| < 1$. The closed range property of the Kohn Laplacian on (S^3, L_f) was studied in [BE] where sufficient and necessary conditions are given. In light of these, it is natural to ask the following question:

Problem 1. On (S^3, L_f), is semi-positivity of the Paneitz operator equivalent to the closed range property of the Kohn Laplacian $\Box_b$.

This question is implicit in [ChChY, CChY]. When the deformation factor f is sufficiently small (in a certain sense), this is true, following from [CChY, ChChY]. The general case seems to be open in both directions.

During the workshop, **Peter Ebenfelt**, **Stephen McKeown**, and **Siqi Fu** discussed the problem. They thought they understood the problem better after these discussions.

Bibliography

[ChChY] Chanillo, Sagun; Chiu, Hung-Lin; Yang, Paul, *Embeddability for 3-dimensional Cauchy-Riemann manifolds and CR Yamabe invariants*, Duke Math. J. **161** (2012), no. 15, 2909?2921.

[CChY] Case, Jeffrey S.; Chanillo, Sagun; Yang, Paul, *The CR Paneitz operator and the stability of CR pluriharmonic functions*, Adv. Math. **287** (2016), 109?122.

[L] Lee, John M, *Pseudo-Einstein structures on CR manifolds*, Amer. J. Math. **110** (1988), no. 1, 157?178.

[GL] Graham, C. Robin; Lee, John M., *Smooth solutions of degenerate Laplacians on strictly pseudoconvex domains*, Duke Math. J. **57** (1988), no. 3, 697?720.

[BE] Burns, Daniel M.; Epstein, Charles L., *Embeddability for three-dimensional CR-manifolds*, J. Amer. Math. Soc. **3** (1990), no. 4, 809?841.

Report on the progress obtained so far on the problem Generalizing Ch. Frances' to the CR setting

The basic idea promoted by the workgroup on Ch. Frances' example relied on the following informal observation: many phenomena, proofs and ideas in conformal geometry were later rediscovered in CR geometry.

The example constructed by Ch. Frances [1] is important in conformal geometry. It shows that in any signature (p, q) with $2 \leq p \leq q$ there exists a compact conformally nonflat manifold such that for this manifold the group of the conformal transformations is essential. The example answers a question asked by M. Gromov and in fact influenced the whole field of parabolic geometry.

To generalize Frances' example to the CR setting, the following strategy was adopted: first **Vladimir Matveev** explained Frances' example in an informal talk. Generalizations were then discussed, and followed by Maple calculations by **Jeffrey Case**, **Sean Curry** and **Vladimir Matveev**. In this way the participants reached a much better understanding of the problem and expect to soon complete a manuscript resolving the question.

Bibliography

[1] Ch. Frances, *About pseudo-Riemannian Lichnerowicz conjecture.* Transform. Groups **20**(2015), no. 4, 1015–1022.

Report on the progress obtained so far on the problem Build a variational theory of chains on a strictly pseudoconvex CR hypersurface in $\mathbb{C}$

^{n} Let $M \subset \mathbb{C}^n$ be a strictly pseudoconvex real hypersurface. Chains are a biholomorphically invariant system of curves transverse to the Levi distribution $H(M)$, discovered by E. Cartan for $n = 2$. Given a smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$, C. Fefferman built a Lorentzian metric F on the product manifold $\partial\Omega \times S^1$ (in connection with the study of the boundary behavior of the Bergman kernel of Ω) and showed that non vertical null geodesics (light rays) of the Lorentzian manifold $(\partial\Omega \times S^1\, ,\, F)$ project on chains of $M = \partial\Omega$.

A construction of Fefferman's metric (due to J.M. Lee) is available for any strictly pseudoconvex CR manifold M, not necessarily embedded and in particular when embedded, not necessarily the boundary of a domain. Fefferman's metric in the J.M. Lee approach is a Lorentzian metric on the total space of the canonical circle bundle $S^1 \to C(M) \xrightarrow{\pi} M$. The problem of building a variational theory of chains as a variational theory of light rays using the *time functional* was posed by C. Le Brun in 1991 and is explicitly stated in the monograph by S. Dragomir and G. Tomassini [DT]. During the worksop we report on, the problem was proposed once again by **Richard Montgomery**.

Sorin Dragomir gave an informal talk explaining Lee's construction of the Feffferman metric $F = F_\theta$ on $C(M)$ in terms of a contact form θ on M. Emphasis was put on the use of the *Graham connection* [a connection 1-form σ on the principal bundle $S^1 \to C(M) \to M$] in horizontally lifting the Reeb vector field $T \in \mathfrak{X}(M)$ of the contact manifold (M, θ) to a vector field $T^\uparrow \in \mathfrak{X}(C(M))$ such that $T^\uparrow - S$ is a globally defined timelike vector field

(S is the tangent to the S^1 action) and hence the synthetic object ($C(M)$, F, $T^{\uparrow} - S$) is a space-time.

Another step towards the understanding of the problem posed consisted in identifying F. Giannoni & A. Masiello & P. Piccione [GMP] as authors of a variational theory for null geodesics, with respect to the time functional, on a globally hyperbolic space-time. F. Giannoni et al.'s work relies on a version of Fermat's principle in general relativity theory and starts from earlier work by V. Perlick [VP] on the time functional. Unfortunately for standard compact examples of CR manifolds, such as the sphere $S^3 \subset \mathbb{C}^2$, the resulting space-time $C(S^3)$ isn't globally hyperbolic. The theory by F. Giannoni et al. appears however to work under conditions more general than stated and a group of participants at the workshop (**Richard Montgomery**, **Howard Jacobowitz**, **Elisabetta Barletta**, and **Sorin Dragomir**) engaged into a research project whose goal is to extend the methods by F. Giannoni et al. to Fefferman space-times with the hope to develop a Morse theory for chains on strictly pseudoconvex real hypersurfaces $M \subset \mathbb{C}^n$ including the known compact examples. There has been also a hint from **Vladimir Matveev** to use a particular Finsler metric (such as Randers' or Kropina's metric) in building the sought after variational theory of null geodesics. This possibility will also be further investigated.

Bibliography

[DT] S. Dragomir & G. Tomassini, *Differential geometry and analysis on CR manifolds*, Progress in Mathematics, Vol. 246, Birkhäuser, Boston-Basel-Berlin, 2006.

[GMP] F. Giannoni & A. Masiello & P. Piccione, *A Morse theory for light rays on stably causal Lorentzian manifolds*, Annales de l'I.H.P., (4)69(1998), 359-412.

[VP] V. Perlick, *On Fermat's principle in General Relativity*, Class. Quantum Grav., 7(1990), 1319-1331.

Report on progress obtained so far on the problem Optimal inequalities of Sobolev type and on related aspects

It is well-known that in the Euclidean space of dimension n the following Sobolev-Gagliardo-Nirenberg inequality holds

$$\left(\int_{\mathbb{R}^n} |u|^{p_n} dx\right)^{\frac{p}{p_n}} \leq C_{n,p} \int_{\mathbb{R}^n} |\nabla u|^p dx; \qquad p_n = \frac{np}{n-p},$$

where $C_{n,p}$ is a positive constant depending only on n and p. Here $n \geq 2$ and $p \in (1, n)$.

When $n \geq 3$ and $p = 2$ functions realizing the best constant have been known for quite some time, see [Aub], [Tal]. These extremals are radial and have been classified using the co-area formula and rearrangement techniques. In [CNV] the optimal functions, still radial, have been classified for general p, using techniques from transport theory. Notice also that in the limit $p \to 1$ one formally obtains the classical isoperimetric inequality.

For the Heisenberg case a similar inequality holds [FS74]. In the n-dimensional Heisenberg group p_n has to be substituted with p_Q, where $Q = 2n+2$ is the *homogeneous dimension* of the space. The optimal constant was found for $p = 2$ in [JLExtr], using a technique inspired from [Oba], but much harder from the technical point of view. In this case indeed, extremals only have partial symmetries and classical methods relying for example on moving plane techniques cannot be applied.

In this last respect, the team has been working on some partial rearrangement techniques that respect the symmetries of the Heisenberg group, in order to work in restricted classes of functions for which a classification might be easier. A first attempt that has been tried was a $\mathbb{Z}_2$ symmetrization in directions orthogonal to horizontal axes. While it looked initially promising, it was realized that in some examples the procedure was not decreasing the Sobolev norm. After this, another attempt was made concerning a proper rearrangement along lines. Some preliminary checks (in cases of smooth functions) looked positive, and the strategy is under current investigation for more general cases.

Another discussion, in a more preliminary stage, regarded existence of entire solutions of supercritical equations in the whole Heisenberg group. For supercritical regimes these equations do not have variational structure, and in [SZ] solutions were found using shooting methods for Ordinary Differential Equations. It would be interesting to show such existence using general tools from Functional Analysis, which might extend to the non-radial setting as well.

Participants to the activity: **L. Capogna**, **S. Chanillo**, **J-H. Cheng**, **G. Citti**, **A. Malchiodi**, **M. Matsumoto**, **J. Tyson**, **C. Wang**, **P. Yang**.

Bibliography

Aub Aubin, Thierry, *Problèmes isopérimétriques et espaces de Sobolev*, J. Differential Geometry, 11, 1976, 4, 573–598, http://projecteuclid.org/euclid.jdg/1214433725,

AubY Aubin, Thierry, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. (9), 55, 1976, 3, 269–296,

CNV D. Cordero-Erausquin, B. Nazaret, C. Villani, *A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities*, Adv. Math., 182, 2004, 2, 307–332, https://doi.org/10.1016/S0001-8708(03)00080-X,

FS74 Folland, G. B. , Stein, E. M., *Estimates for the* $\bar{\partial}_b$ *complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math., 27, 1974, 429–522, https://doi.org/10.1002/cpa.3160270403,

JLExtr Jerison, David, Lee, John M., *Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem*, J. Amer. Math. Soc., 1, 1988, 1, 1–13, https://doi.org/10.2307/1990964,

Oba Obata, Morio, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan, 14, 1962, 333–340, https://doi.org/10.2969/jmsj/01430333,

SZ Serrin, James, Zou, Henghui, *Existence and nonexistence results for ground states of quasilinear elliptic equations*, Arch. Rational Mech. Anal., 121, 1992, 2, 101–130, https://doi.org/10.1007/BF00375415,

Tal Talenti, Giorgio, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4), 110, 1976, 353–372, https://doi.org/10.1007/BF02418013,