Projective modules and A1-homotopy theory

organized by
Aravind Asok, Jean Fasel, and Satyagopal Mandal

Workshop Summary

This workshop brought together people working in and around the classical theory of projective modules and people working in $\mathbb{A}^1$-homotopy theory. Introductory lectures providing some common ground for the participants were given by Röndigs ($\mathbb{A}^1$-homotopy theory), Nori (Euler class groups), Fasel (Chow-Witt groups), Mandal (Complete intersections), Haution (Motivic cohomology operations), Williams (Classical obstruction theory). Research talks were given by Stavrova ($\mathbb{A}^1$-connected component sheaves of linear algebraic groups), Sawant/Balwe (Connected components of the singular construction) and Bhatwadekar (Recent progress on the biregular cancellation problem in positive characteristic).

Subsequent discussion focused on the problem of comparison of Euler class groups, as well as some problems raised in the discussion, which will be discussed in greater detail below.

- Nori gave a new definition of the Euler class group and proposed the problem of analyzing its relationship with the definition given by Bhatwadekar-Sridharan. That the definition could be generalized to the situation of $\mathbb{A}^1$-homotopy was observed by Fasel-Haesemeyer.
- Balmer observed that the Euler class in Chow-Witt theory was defined using constructions that looked fairly categorical, and he proposed the problem of defining an Euler class obstruction in the context of his tensor-triangulated geometry. Nori observed that, in fact, one could make a purely tensor triangulated description of the Euler class. It was also observed that, in the general context, the relationship between the vanishing of the Euler class and existence of a splitting of the appropriate form is more complicated.
- Stavrova asked the question: is $\pi_0^{\mathbb{A}^1}(G)$ abelian in relation to a question of Colliot-Thélène regarding whether the group of R-equivalence classes $G(k)/R$ is abelian. Combining results of Balwe/Sawant/Hogadi and an observation of Nori, it was observed that $\pi_0^{\mathbb{A}^1}(G)$ is *not* abelian for $k = \mathbb{R}$ and G an anisotropic k-group. In contrast, in the case where G is isotropic over a field k (say having characteristic 0), $\pi_0^{\mathbb{A}^1}(G) = G(k)/R$. It is expected that some combination of the above authors will discuss this example in future work.
- Fasel, Haesemeyer, Jacobs, and Sane studied the question if k is an arbitrary field, and X is a smooth affine k-variety of dimension $2d + 1$, does a vector bundle of rank $2d + 1$ with trivial top Chern class split off a free rank 1 summand. They attempted to construct a counterexample to this problem by studying certain universal hypersurfaces.

- Wickelgren proposed the problem of studying an analog of the Serre spectral sequence in $\mathbb{A}^1$-homology. The existence of such a spectral sequence in topology is a fundamental computational tool, and, barring issues about computability of $\mathbb{A}^1$-homology would no doubt have fantastic implications in $\mathbb{A}^1$-homotopy theory. Williams, Wickelgren and Röndigs discussed this and made some progress in some special cases, specifically in the case of G-torsors under linear algebraic groups.
- Nori proposed a problem about when a class in K_0-theory of a smooth affine scheme X of dimension d admitted a representative by a vector bundle of rank $d-1$. This problem was studied in detail, and it was observed to be equivalent, in the case d odd, to the problem about splitting vector bundles of odd rank as above. It was also observed that in case d is even, the top Chern class is the precise obstruction, following work of Sridharan and Bhatwadekar.

The problem on splitting free rank 1 summands from top rank bundles on odd-dimensional varieties attracted a lot of attention at the workshop, and it is expected that some subset of the participants who were involved in its study will continue to investigate this problem. It was also observed that one of the problems posed on the announcement, regarding, e.g., complete intersections or Murthy's question, are amenable to cohomological study. It is expected that participants of the workshop will continue to work on these problems.