Positivity of cycles

organized by
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Workshop Summary

The workshop studied the positivity of cycles of higher codimension. The primary focus was the study of "positive cones": pseudo-effective and movable cones of cycles, and nef cones (and related cones) in the dual space. Several other topics were discussed in less detail: the relationship of numerical positivity with other geometric and cohomological properties, the positivity of 0-cycles under rational equivalence, and the Fulton-Hansen property. Most of the participants had used these concepts in their own work, but had perhaps not systematically studied these structures. Thus the goals of the workshop were two-fold. First, to lay a foundation of common concepts, examples, and definitions, to establish a working background for the participants. Second, to set directions for future study in this young but growing field.

The workshop closely followed the AIM format. In the first morning, the organizers gave introductory lectures on the topics of the workshop. In succeeding mornings, two or three participants would report on previous work relevant to the workshop. The organizers made a particular effort to include graduate students as presenters.

After holding an open problem session on Monday afternoon, the afternoons were devoted to working groups. After some discussion, most groups decided to focus on examples; indeed, many of the open problems centered on finding useful counterexamples for various questions. In addition, examples are amenable to computation over the course of a week, while most theoretical questions seemed to require a longer period of study.

Group sessions

Projective bundles over surfaces:

One group focused on the computation of pseudo-effective cones for projective bundles over $\mathbb{P}^2$. Such examples should give a rich source of test cases for conjectures about higher-codimension cycles, while remaining amenable to computation. Some of the key problems are:

- Must such cones be simplicial? Finitely generated?
- Find examples where the pseudo-effective cone is not generated by classes on a section and classes on the restriction to a line.
- Do projective bundles provide a counterexample to the Strong Conjecture of Debarre-Jiang-Voisin?
- Do stable or exceptional bundles exhibit special properties with respect to pseudo-effective cones?
- Are there examples where the nef cones are not contained in the pseudo-effective cones?

First, the group addressed the case of split bundles. Since such bundles admit a $\mathbb{C}^*$-action, the computation can easily be accomplished by induction. A similar situation arises for exceptional, or more generally for homogeneous vector bundles.

The group then turned to several well-known examples of non-split rank 2 bundles, such as bundles arising from the Serre construction or from monads. Already such examples exhibit interesting phenomena. For example, for some bundles the computation of the pseudo-effective cone is equivalent to the Nagata conjecture on blow-ups of the projective plane.

The next case under consideration was stable bundles. The participants made a number of natural guesses, but further work is required to prove their expectations. The group plans to continue working on this project and expect to obtain at least one paper arising from their computations.

Moduli spaces of curves:

One group studied the pseudo-effective cones of moduli spaces of pointed rational curves. There are a host of well-known conjectures and questions concerning the behavior of subvarieties on such spaces, most of which are completely open in the symmetric case. The group focused on the first interesting example $\overline{M}_{0,7}^{S_7}$. Since the space of surfaces on this fourfold is only three-dimensional, the cone $\overline{Eff}_2(\overline{M}_{0,7}^{S_7})$ is a natural and important example. While the intersection theory of this space is well-understood, in practice computations are rather involved.

The main question under consideration was whether $\overline{M}_{0,7}^{S_7}$ exhibited any pseudo-effective classes outside the cone spanned by boundary strata. The group tested several natural candidates, but have not yet found any such classes. They plan to continue this work by analyzing other examples and by trying to prove upper bounds on the pseudo-effective cone, with the intent of writing a paper as the work progresses.

An additional outcome of the workshop was a computational tool developed to facilitate computations for the space. This tool already managed to catch an error in earlier published papers, and is likely to be useful to the mathematical community moving forward.

Blow-ups and hypersurfaces:

One group studied pseudo-effective cones in a variety of different circumstances. The goal was to identify new cases of unexpected or subtle behavior, e.g. nef classes which are not pseudo-effective. The challenge was to navigate between examples that are easily computable but exhibit uninteresting behavior, and examples that seem interesting but are too difficult to compute.

First, the group studied classes on blow-ups of projective space. Previous work has focused on point blow-ups. This group instead focused on blow-ups of lines, blow-ups of curves, and blow-ups of general complete intersections. In the first two cases, the group found that (at least in small cases) the behavior of pseudo-effective cones reflected interesting classical geometry. As the degree or number of blow-ups grows, the computation appear to become intractable. In contrast, the blow-up along a single complete intersection seems to have very regular behavior, regardless of the degree or codimension. The group wrote out the first few cases and plans to identify and prove the pattern suggested by them.

Second, the group studied hypersurfaces in $\mathbb{P}^m \times \mathbb{P}^n$ and in $G(2, n)$. While the numerical spaces are still not too large, such computations seem tricky in practice. In high degree, the computations can be accomplished by building off of previous work concerning the pseudo-effective and nef cones of divisors. The group identified $(2, 2)$-divisors in $\mathbb{P}^2 \times \mathbb{P}^3$ and $\mathbb{P}^2 \times \mathbb{P}^4$ as examples worthy of further study, since they identified effective cycles outside of the "obvious ones".

The group plans to continue their analysis and to write a paper on the results.

Estimating mobility counts:

One group considered the problem of estimating mobility counts on varieties with a rich structure, such as $\mathbb{P}^2 \times \mathbb{P}^2$ or $G(2, n)$. The group made a number of computations, but failed to find any constructions improving upon previously known bounds. After some discussion, the group decided to stop work on the first day and to focus on problems more suitable for a week-long workshop.