Rigidity and polyhedral combinatorics

organized by
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Workshop Summary

Monday afternoon's discussion, led by Ivan Izmestiev, concerned various variations on Alexandrov's theorem and their proofs by variational principles. Alexandrov's theorem states that any intrinsically convex surface has a realization in 3-space as convex polytope, or a doubly covered convex polygon. Izmestiev has an algorithm that can do this realization, whereas Alexandrov's original proof was not really constructive.

Throughout the week, there was a thread of sessions about algebraic dependencies among face volumes of polyhedra, especially simplices. Monday's discussion, led by Idzhad Sabitov, Bob Connelly, and Ezra Miller, reviewed the algebraic background, in the specific contexts of volume polynomials for 3-polytopes, Sabitov's theorem, and higher genus surfaces in 3-space. Sabitov's result can be stated as saying that the volume bounded be a triangulated oriented surface is integral over the ring generated by the edge lengths. In other words, there is a monic polynomial satisfied by the volume, where the coefficients of the polynomial are themselves polynomials in the squares of the edge lengths. The question is whether there are other geometric quantities that have a similar integrality relation. Tuesday's discussion, led by Miller, constructed a faulty—but nonetheless not entirely unenlightening—argument in favor of integrality for k-volumes of simplices over the edge lengths. Wednesday's discussion, led by Igor Rivin, concerned the application of singular values of Jacobians to prove generic integrality of $(n-2)$-volumes. Thursday's discussion, led by Ezra Miller, Ileana Streinu, and Eran Nevo, proved the equivalence of k-volumes with $(n-k)$-volumes in simplices, as long as the n-volume is nonzero, using duality in exterior algebras. In the end, participants in these discussions felt they understood the integrality issue for simplices pretty well, although what happens in the zero full volume case is still an interesting mystery.

The Tuesday afternoon discussion led by Dylan Thurston concerned generic global rigidity. Global rigidity is a basic property of bar frameworks in Euclidean d-space. It says that the configuration of a framework is determined uniquely, up to congruence in d-space, by the lengths of the edges of the graph that define the framework. Determination of global rigidity for an arbitrary graph is known to be equivalent to a large number of known difficult problems, and so probably it not reasonable to expect to be able to compute it in general. On the other hand, it seems to be reasonable to expect to be able to compute an framework's global rigidity when the configuration is generic. In this case, Connelly proved that if the rank of an associated matrix, called the stress matrix, is maximal, it is sufficient to assure global rigidity in any dimension. Dylan Thurston and his co-authors have recently shown that this criterion is necessary as well as sufficient. This provides a complete characterization of generic global rigidity.

On Wednesday afternoon, we had a nice session on unfolding. About six people (give or take two) were present. Alexey Tarasov presented his counterexample to the following strengthened form of the so-called Dürer conjecture: if a convex 3-polytope is given, with geodesics drawn between the vertices to form a 3-connected planar graph having convex faces, then after cutting the surface along some spanning tree of this graph, the remaining surface unfolds flat into the plane. (The original Dürer conjecture is the case where the geodesics are the natural edges of the convex polytope.) Tarasov showed that there exists a polytope and such a graph on its vertices such that no spanning tree produces a foldout without overlap. The technically involved counterexample was presented in great detail and received a lengthy discussion. Although the given construction is very far from being the edge set of a convex polytope, there was some support for the possibility that this approach might pave the way for disproving the original Dürer conjecture.

On Thursday we had a session on A.D. Alexandrov's theorems and pointed spherical tilings. The official session was combined with multiple unofficial discussions. First, Gaiane Panina gave a presentation of her recently obtained counterexamples to A.D. Alexandrov's conjecture, which posits that given a smooth compact convex body K in R^3, if a constant C separates (non-strictly) the principal curvatures at every point of its boundary, then K is a ball. Panina indicated the relationship to combinatorial rigidity theory, namely to the theory of self-stressed pointed graphs embedded in the 2-sphere. Sergei Tabachnikov raised the following natural question: how does this relate (or generalize) the Arnold's 4-vertex theorem, Möbius theorem, or tennis ball Arnold's theorem. The conclusion was that there is a relationship, but not a straightforward one.

Thursday afternoon, Hellmuth Stachel and Idzhad Sabitov led a session on higher-order rigidity. First-order rigidity of bar-and-joint frameworks, or equivalently infinitesimal rigidity, which is sufficient but not necessary for rigidity is a natural first test. Similarly, second-order rigidity, which is also sufficient but not necessary for rigidity, is a natural second test. Unfortunately, the natural extension of the formulas to third-order rigidity is not satisfactory, since there are examples of frameworks that are third-order rigid by the naive definition, but they are not rigid. There were some possible candidates for a good definition of higher-order rigidity, but they are complicated, and have some undesirable drawbacks.

Early in the workshop, there was an intriguing discussion about the Stoker conjecture, which says that the facial angles of a non-degenerate 3-dimensional polytope are uniquely determined by its dihedral angles. There are three versions of the conjecture: infinitesimal, local, and global. The conjecture can be reformulated as a rigidity conjecture of positively stressed frameworks embedded in the 2-sphere. The infinitesimal version was known to be true, while the other versions puzzled the participants a lot. Walter Whiteley hinted (in the open problems list) that he had some tools to disprove the conjecture. Gaiane Panina also had some approaches for disproving it, together with some examples. But Rafe Mazzeo, visiting from Stanford University for the day, led a large session on Wednesday afternoon to give the ideas of his topological proof of the local version of the Stoker conjecture.

A number of times during the week, interactions of rigidity theory with algebraic geometry arose in a variety of contexts. By Friday afternoon, as the workshop was winding down, there was general agreement that it would be good to get everybody on the same page regarding the geometry of schemes, particularly nonreduced structures, as they pertain to rigidity. An hour of discussion of this topic was led by Ezra Miller.