Phase transitions in randomized computational problems

organized by
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Workshop Summary

1. Introduction; aims and scope

This workshop was devoted to the study of random constraint satisfaction problems (CSPs), with an emphasis on threshold phenomena and related algorithmic challenges. Heuristic methods of statistical physics predict a rich phase diagram for many problems of this type. In recent years, building on the physics insight, some of these predictions have been rigorously proved. Our aim in organizing this workshop was to promote discussions and collaborations that could lead to further progress in these directions, focusing on the following topics:

(1) *Solution geometry.* The solution space of a random CSP is a random subset of the space of all possible variable assignments. There are many interesting open questions concerning the typical geometric properties of this space. For example, a large class of sparse random CSPs (k-SAT, coloring, independent set) is expected to exhibit "one-step replica symmetry breaking" (1RSB) – in a certain regime, it is conjectured that the solution space consists (mostly) of a large bounded number of connected components, with relative weights distributed as a Poisson–Dirichlet process. Some manifestations of this prediction have been proved in recent years, but the Poisson–Dirichlet picture remains an interesting open question.

(2) *Full RSB in sparse graphs.* For some well-known models, the solution geometry is believed to be much more complicated than 1RSB. For example, the MAX-CUT problem on sparse random graphs is believed to exhibit full RSB (infinitely many levels of replica symmetry breaking). In comparison there is a relatively complete understanding of the Sherrington–Kirkpatrick model, which also exhibits full RSB but in the setting of dense graphs. In the sparse case there has been some recent progress, partly based on comparisons with the dense regime, but many fundamental questions remain open.

(3) *Algorithmic challenges.* It is conjectured that the geometry of the solution space is related to algorithmic challenges. However, it seems currently out of reach to establish hardness of random instances against general algorithms. The sum-of-squares (SOS) hierarchy provides one possible framework for studying this question: it defines a powerful but restricted class of algorithms, and there have been recent successes in establishing SOS hardness results for random CSPs. These methods also exhibit phase transitions which are expected to be of interest.

The participants of the workshop came from quite diverse backgrounds, with interests ranging from statistical physics (including neural networks and graphical inference problems) to

mathematics (including probabilistic combinatorics, spin glass theory, and random matrices) to theoretical computer science (including algorithmic design and the sum-of-squares hierarchy).

2. Summary of workshop activities

In the morning sessions we had ten very nice talks contributed by workshop participants, which we briefly summarize below. We have loosely assigned the talks to three categories, but we feel there were many connections between different talks that could be explored in future:

(1) *Sparse random CSPs.* In the first opening talk, Andrea Montanari surveyed the statistical physics heuristics for belief propagation and replica symmetry breaking. His talk highlighted several outstanding open problems in the area, especially in the context of random CSPs. Allan Sly reviewed some mathematical progress for sparse random CSPs with one-step replica symmetry breaking, including satisfiability thresholds and free energy results. Will Perkins presented some recent work on information-theoretic thresholds for planted random CSPs (e.g. for some cases of the stochastic block model), which leverage some regularity-type results concerning measures on cubes.

(2) *Spin glasses.* Aukosh Jagannath presented an overview of recent mathematical progress for mean-field (fully connected) spin glass models. The talk surveyed numerous works by several people; this includes recent results on temperature and disorder chaos, as well as some universality-type results, relating fully-connected models to their sparse counterparts at large but bounded connectivity. We also had two talks focusing on the Gibbs measure geometry: Gérard Ben Arous presented recent work on a large deviations principle (related to the 2D Guerra–Talagrand bound obtained by Panchenko) for two-replica dynamics in the p-spin Ising model, which leads to new results on free energy barriers and slow mixing of Glauber dynamics. Eliran Subag presented some of his work which gives a quite detailed characterization of the landscape of the pure p-spin spherical model at low temperature, including notably the discovery that the model lacks temperature chaos.

(3) *Algorithmic challenges.* In the second opening talk, Uri Feige discussed algorithmic challenges for random CSPs. In particular, he suggested some motivations for moving away from the pure random models, to semi-random models that include an adversarial component; his talk mentioned several nice open problems in this general direction. David Gamarnik presented works relating the geometry of the solution space to algorithmic barriers; there is some connection to the free energy barriers in the talk of Gérard Ben Arous. Ryan O'Donnell gave an overview lecture on the sum-of-squares (SOS) framework, and surveyed recent work on the SOS threshold for strong refutation of random CSPs. Tselil Schramm presented in detail one of these works, which gives an SOS-based algorithm for strongly refuting random CSPs up to $n^{k/2}$ clauses.

The afternoon sessions were devoted to discussions of open problems revolving around the following themes:

(1) Thresholds for SOS/SDP methods

(2) Thresholds in inference problems with restricted computation
(3) Semi-random problems (combining pure random and adversarial components)
(4) Problems averse to spectral methods
(5) Replica symmetric mean-field models

The discussion groups seemed to be fairly well mixed in terms of individual research interests, and several participants formed new collaborations as a result. Some groups made substantial progress on their chosen problem, which is being continued in ongoing work after the workshop's end.