

# MANIFOLDS WITH NONNEGATIVE SECTIONAL CURVATURE

organized by  
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## Workshop Summary

In the past few years, the study of Riemannian manifolds with nonnegative and positive curvature has been reinvigorated by breakthroughs and by new connections to other topics, including Ricci flow and Alexandrov Geometry. Our workshop brought together experts and newcomers to the field, including 5 graduate students and researchers representing a diverse range of sub-specialties. Our goal was to discuss future directions for the field and to initiate progress solving significant open problems.

We had on average two talks every morning. These talks were primarily surveys emphasizing open problems and possible future directions for continued progress. The talks were roughly divided between the following three sub-topics, which we identified as key to continued progress in the field:

- (1) Riemannian submersions and group actions in nonnegative curvature
- (2) Alexandrov Geometry and Collapse
- (3) Ricci flow

On Monday afternoon, all participants gathered to list and discuss open problems related to nonnegative curvature. The lively discussion lasted almost 3 hours, and resulted in a preliminary list of about 30 open problems. Many of these problems prompted interesting discussions. Participants continued to add problems to this list during subsequent days of the workshop. The list will continue to evolve, and has the potential to become a useful resource for future researchers in the field.

On Tuesday afternoon, we divided the workshop participants into three groups, corresponding to the three sub-topics enumerated above. This subdivision remained roughly constant through the remainder of the week, with a few participants choosing to float between the groups. The groups learned of each other's activities informally each evening during happy hour, and more formally through group reports on Friday afternoon.

The first group explored Riemannian submersions and group actions. This group was the largest, and its members decided to further subdivide. They began by brainstorming possible problems to attack in smaller subgroups. Before splitting up, they scheduled mini-talks to explain some recent unpublished work. These mini-talks helped participants (especially newcomers to the field) decide which subgroup they felt best equipped to join. One subgroup formed to begin classifying the Riemannian submersions from a compact Lie group with a bi-invariant metric. This subgroup quickly discovered an interesting non-homogeneous example, which contradicts the naive conjecture that all such submersions are bi-quotient submersions. This subgroup then spent most of the remaining time considering the case of totally geodesic fibers. This collaboration will likely lead to a paper in the coming months. A second subgroup formed to bound the dimension of a torus acting freely on a manifold

with nonnegative curvature, and to consider related problems. The third (largest) subgroup investigated cohomogeneity-one manifolds with nonnegative curvature. They discussed this topic from several angles. They considered cohomogeneity one manifolds with a totally geodesic principle orbit, and came to believe that the classification of such spaces is within reach. A collaboration on this problem will continue and probably lead to the complete solution in the near future. They also considered obstructions to metrics of nonnegative curvature and smoothness conditions for cohomogeneity one actions. Finally, some of the members of this subgroup considered topological aspect of cohomogeneity-one manifolds, including topological invariants of known and candidate examples and the problem of finding cohomogeneity-one manifolds which are topologically interesting, and for which the problem of constructing new metrics with nonnegative curvature or obstructions should thus be investigated.

The second group explored Alexandrov geometry and collapse. This group began by generating a list of about 20 interesting open problems. They then chose three of these problems to explore in more depth. The first of these problems was to extend (the dual version of) Wilking's connectivity lemma to Alexandrov spaces. The group mapped out a proposal involving Morse functions to solve this problem. This work will hopefully lead to a collaborative solution in the near future. The second problem was to discover topological properties of an Alexandrov space which sits at the top of a finite tower of fiber bundles. The third problem was the conjecture that all manifolds with almost nonnegative curvature are rationally elliptic. The group discussed a rough strategy for how a proof by induction on dimension might go. One important step in such a proof would be to show that any Alexandrov space which collapses to a point also admits nontrivial collapse. In exploring this issue, the group constructed an essentially complete proof that the torus does not collapse to an interval.

The third group studied Ricci flow. Recent progress in the applications of the Ricci flow to manifolds with positive curvature operator, positive isotropic curvature and manifolds with  $1/4$  pinching were discussed in the morning survey talks. The Ricci flow subsection gave a simple proof of Tachibana's theorem that an Einstein metric with positive curvature operator is a space form. Two of the participants generalized recent work by Boehm and Wilking on even dimensional manifolds with small Weyl tensor to the odd dimensional case and gave a simple proof of the algebraic part needed in the proof of the weakly  $1/4$  pinching theorem. Furthermore, existence and stability of singularity models and the nonexistence of noncompact 3 dimensional shrinkers was discussed. They also discussed the problem of ruling out noncompact gradient shrinking solitons with positive curvature operator, or more generally classify the gradient shrinking solitons with certain positivity of the curvature.

The participants were almost unanimous in feeling that the workshop was successful. One participant stated that "all conferences should be structured this way". Of course one should add that this is only possible with a narrowly focused research area. The afternoon group-work varied between brainstorming ideas for solving very difficult open problems and solving easier problems. Work at either extreme of this spectrum was felt to be productive and meaningful. Often the groups continued working past the 5:00 beginning of happy hour (even the most beer-loving of the groups) which demonstrates the energy that the group members felt. We expect that new collaborations will develop as a result of the workshop. Further, participants are returning home with new ideas that could shape the long term development of the field in less tangible ways.

We are very thankful for the generous support of the AIM. We appreciate the guidance and hard work of the AIM staff in helping us conduct a successful workshop.