Non-Hermitian quantum mechanics and symplectic geometry

organized by
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Workshop Summary

This workshop brought together 19 experts from the three areas of symplectic geometry, quantum physics, and spectral analysis. The motivation for the workshop were several recent developments, all of which connected to *complexifications* of different kinds. Specifically, similar structures were identified in the evolution of Gaussian wave packets (coherent states) in quantum systems with non-Hermitian Hamiltonians in the semiclassical limit, the complexification of the group of Hamiltonian symplectomorphisms on a Kähler manifold, and the spectral theory of non-selfadjoint operators and semigroups generated by them. Inspired by these specific connections between the three areas, the workshop was devoted to further exploring the interconnections and attempting to use these to attack some pressing mathematical problems in the following areas:

- Physical problems leading to non-Hermitian quantum Hamiltonians;
- Spectral problems for non-Hermitian operators;
- Semi-classical aspects of non-Hermitian quantum mechanics; and
- Non-Hermitian evolution of wave packets and the complexification of the group of Hamiltonian diffeomorphisms of a symplectic manifold.

The structure of the workshop followed the usual AIM procedures, with two didactic talks each morning, and afternoon group sessions devoted to work on a number of problems that were identified in a plenum on Monday afternoon.

Morning talks

On **Monday July 16**, Eva-Maria Graefe gave an overview on non-Hermitian Hamiltonians in quantum physics. Lindblad dynamics, complex potentials in beam propagation in quantum waveguides, and PT-symmetric Hamiltonians were emphasized. Eva presented a result from a collaboration with Roman Schubert, regarding the propagation of Gaussian wave packets under non-Hermitian Hamiltonians. In the semiclassical limit the centers of these Gaussian states move according to a new type of phase-space motion, coupled to an equation of motion for the covariance matrix of the Gaussian, which can be interpreted as a phase-space metric. Similar structures had been discovered in the context of geometric quantization, the topic of Tuesdays morning talks. Concluding the Monday morning session, Michael Hitrik then spoke about semi-classical spectral theory for non-Hermitian operators P, such as the Schrödinger operator with a complex potential. A striking phenomenon in the non-selfadjoint case is the existence of pseudo-eigenvalues, complex numbers z away from the spectrum such that the norm of the resolvent $(P - zI)^{-1}$ blows up as Planck's constant tends to zero. The talk began with an exposition of earlier results that establish conditions for a complex number to be in the semi-classical pseudo-spectrum, defined as

$\{z \in \mathbb{C}\ ;\ \|(P - zI)^{-1}\| \geq C_N^{-1}\hbar^{-N}, \forall N \in \mathbf{N}\}$. It is known that under mild conditions, the semi-classical pseudo-spectrum of an $\hbar$-pseudodifferential operator is contained in the closure Σ of the image of its principal symbol, and for Schrödinger operators with analytic potentials, say, the interior of said closure is in the semi-classical pseudo-spectrum. These results were reviewed, as well as results on the behavior of the resolvent near the boundary of Σ. Then more recent results were presented on non-selfadjoint perturbations of a self-adjoint operator, in the analytic case. By complex deformations of the real phase space $\mathbb{R}^{2n}$ into specially constructed I-Lagrangian submanifolds of $\mathbb{C}^{2n}$, one can attempt to reduce the pseudo-spectrum, and thereby study the spectrum more efficiently.

The talks on **Tuesday July 17** were devoted to the notion of complex Hamiltonian symplectomorphisms of a Kähler manifold and its relation with geodesics on the space of Kähler potentials. Alejandro Uribe spoke on Donaldson's complexification of the group of Hamiltonian symplectomorphisms and quantum mechanics. Assuming that the Kähler manifold is real analytic, Alejandro explained how to construct geometrically the exponential map of a real analytic complex valued Hamiltonian on the manifold. A related approach to the complexification of Hamiltonian flows and corresponding geodesics on the space of Kähler metrics, motivated by questions of quantum field theory, was discussed in the talk by José Mourão. For cotangent bundles of symmetric spaces of compact type, U/K, the infinite geodesics associated with convex functions of the moment map link, at infinite geodesic time, Kähler polarizations with mixed Kirwin-Wu polarizations. The corresponding real directions have associated Bohr-Sommerfeld leaves related with K-spherical representations. Generalized coherent state transforms correspond to holomorphic sections degenerating (at infinite geodesic time) to delta functions on the real directions and holomorphic sections on the complex directions.

On **Wednesday July 18**, Maciej Zworski spoke on internal waves in fluids and spectral theory of 0th order operators. The starting point here has been the fascinating connection, recently found by Colin de Verdière and Saint-Raymond, between modeling of internal waves in stratified fluids and spectral theory of 0th order pseudodifferential operators on compact manifolds. In his talk, Maciej explained how a version of their results follows from the radial estimates for pseudodifferential operators, avoiding the use of Mourre theory, normal forms, and Fourier integral operators. Numerical simulations were also provided. Leonid Polterovich then gave a talk on quantum footprints of symplectic rigidity. Proceeding in the context of the Berezin-Toeplitz quantization, Leonid discussed the notion of the quantum speed limit, a quantum counterpart of the symplectic displacement energy, a fundamental symplectic invariant. Discussed were also Poisson bracket invariants of open covers of a closed symplectic manifold by displaceable sets.

On **Thursday July 19**, Brian Hall introduced the Segal-Bargmann transform on compact Lie groups, defined by the analytic continuation of the forward heat operator, from the Lie group to its complexification. The principal focus of the talk was on the case when the Lie group in question is the group of $N \times N$ unitary matrices, in the large N limit. In his talk, Robert Littlejohn discussed the topic of multisymplectic geometry as an approach to classical and quantum mechanics with constraints, using both the Lagrangian and the Hamiltonian formalism.

On **Friday July 20**, László Lempert discussed in his talk the space of Kähler metrics on a compact complex manifold X, equipped with a Kähler form. The corresponding space of

Kähler potentials $\subset C^\infty(X)$ can be regarded as a Fréchet manifold, and it can be given the structure of an infinite dimensional Riemannian manifold, when equipped with the Mabuchi metric. Geodesics and local isometries in the space of Kähler potentials were discussed in the talk. The last presentation in the series of the morning talks was delivered by San Vũ Ngọc, who spoke on Bohr-Sommerfeld quantization conditions and integrable systems. After reviewing the notions of a classically completely integrable system and of action-angle variables in a neighborhood of a regular Lagrangian torus, San proceeded to discuss quantum completely integrable Hamiltonians and the question of computing the joint spectrum of a quantum completely integrable system. In the one-dimensional case, this reduces to the celebrated Bohr-Sommerfeld quantization condition near a regular energy level of the classical Hamiltonian. The talk was concluded by the discussion of one-dimensional Bohr-Sommerfeld quantization rules near singular levels, addressing both the case of elliptic and hyperbolic critical points.

Afternoon working groups

The following specific problems were discussed during group work in the afternoon sessions. They grew out of a much longer original list of problems suggested by the participants on Monday.

(1) Study the location of eigenvalues for non-selfadjoint quantum Hamiltonians. Although a lot is known (semi-classically) about the pseudo-spectrum, much less is known about the spectrum itself. A first problem was to study the case of differential operators with analytic coefficients in one degree of freedom, for example, the following non-Hermitian Hamiltonian:

$$H = \omega a^\dagger a + \beta(a + a^\dagger) + \chi(a^\dagger a)^2 - i\gamma a^\dagger a. \tag{1}$$

Here $a^\dagger$ and a are the one-dimensional creation and annihilation operators, respectively, and ω, β, χ, and γ are real parameters. More specifically, the questions were:

  (a) Is there a link to the Bohr-Sommerfeld rules (well established in one dimension in the self-adjoint case)?

  (b) In examples (both rigorous and numerical) the eigenvalues seem to lie on unions of arcs of curves. Is there a general theorem about this? What are the curves?

(2) Study the propagation of coherent states under non-Hermitian evolution. More specifically, working in the Bargmann representation, the problem is to show that the center of a propagated (and normalized) coherent state under a non-Hermitian quantum Hamiltonian follows the exponential of the complexification of the group of Hamiltonian symplectomorphisms. This has been proved by Graefe and Schubert for quadratic Hamiltonians, but it is not yet known in greater generality.

(3) The previous problem is related to the general problem of approximating the quantum propagator, in the semi-classical limit. In this context some work was done on the problem of extending the construction of the Herman-Kluk propagator (a parametrix for the propagator in the Hermitian case given by a Fourier integral operator with a global complex-valued phase) from Euclidean space to more general phase spaces, such as the cotangent bundle of a compact Lie group. The latter has a Kähler structure, and Hall's extension of the Segal-Bargmann transform to this setting constructs the coherent states necessary for the Herman-Kluk ansatz.

(4) Continuing with the theme of complex structures on cotangent bundles, investigate the polynomial growth bounds for holomorphic extensions of the eigenfunctions of the Laplacian on a real-analytic compact Riemannian manifold on the Grauert tubes, and the relationship of these extensions with the radius of the maximal Grauert tube.
(5) Study the quantization of symplectic quasi-states, and possible extensions to complex-valued Hamiltonians.

Some initial progress was achieved on all of the problems listed above during the group work. To mention one such development, surprisingly problems (1) and (5) were combined by considering classical Hamiltonians of the form $H = f + i\epsilon g$, where f and g are real-valued and ϵ is a small parameter. The Reeb graph of f (used in the construction of a particular symplectic quasi-state) can be embedded into the complex plane using f and the average of g along trajectories of f. Using a local "averaging method" one should be able to prove that the spectrum of the quantization of H should be close to the embedded graph. A second development, this one regarding problem (2), was the beginning of a study a non-linear (and non-local) Schrödinger equation that is norm-preserving and may be suitable to study the propagation of singularities in the non-Hermitian setting.

The afternoon discussions provided a crucial initial impetus to continue studying these and other related questions. We are optimistic that these problems will be pursued by some of the workshop participants, and feel that the workshop was successful in identifying new fruitful research directions and creating opportunities for new collaborations.

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