Moduli problems beyond geometric invariant theory

organized by
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Workshop Summary

The “intrinsic” approach to studying a moduli problem in algebraic geometry is to represent the moduli problem as an algebraic stack. The structure of the algebraic stacks that arise in examples can be quite complicated. In the last few years, new general tools have been developed that allow one to identify a notion of “semi-stable” points in an algebraic stack, to construct a moduli space for the semistable points, and to construct a canonical stratification of the unstable locus into pieces that are easier to study.

The goal of this workshop was to bring together experts working on different kinds of moduli problems in algebraic geometry, and to attempt to apply these new foundational tools in these areas. The main examples discussed were: 1) alternative compactifications for the moduli of curves, 2) moduli of line bundles and principal G-bundles on nodal curves and higher dimensional varieties, 3) moduli of higher dimensional varieties and pairs. Another focus of the workshop was to find connections between the theory of non-reductive GIT and algebraic stacks.

The first two talks (on the first day) gave an overview of this intrinsic approach to stability and moduli spaces. In the remaining talks, experts presented on each of the moduli problems mentioned above, and on the theory of non-reductive GIT.

Talks

(1) Daniel Halpern-Leistner, *A user’s guide to moduli problems beyond GIT.*
(2) Jochen Heinloth, *A worked out example:* Bun_G.
(3) Chenyang Xu, *K-moduli space of Fano varieties: construction and properties*
(4) Frances Kirwan, *Non-reductive GIT*
(5) Maksym Fedorchuk, *Something about the MMP for* M_g.
(6) Kristin DeVleming, *Computations in K-stability and K-moduli*
(7) David Rydh, *Almost good moduli spaces* (relevant to non-reductive GIT)
(8) V. Balaji, *Torsors on semistable curves and degenerations*
(9) Toms Gmez, *Singular principal G-bundles on higher dimensional varieties*
(10) Sndor Kovcs, *Something about families of varieties in higher dimensions*

Problems discussed

On Monday afternoon, there was a problem session moderated by Max Lieblich. Around 20 research problems were proposed. We summarize below the problems that were discussed in working groups, and some progress that was made.

Stability for curves.

The standard compactification of the moduli of curves is the Deligne-Mumford compactification, but there are several other interesting compactifications parameterizing curves with different kinds of singularities. The main problem here is to systematically describe these different notions of stability, and to understand the wall-crossing behavior as the allowed singularity types varies. The first basic question is for these known moduli stacks of singular curves, can one directly verify that the stack is Θ-reductive and S-complete?

The more ambitious question is whether there is some large (unbounded) stack of curves with very mild restrictions on singularities such that you can recover known stability conditions using Θ-stability on this stack. Also, are each of these semistable loci the open piece of a Theta-stratification, analogous to the Harder-Narasimhan stratification of the moduli of vector bundles on a curve.

Initial progress: one group found an example that shows that the stack of nodal curves is NOT Θ-reductive or S-complete. The problem results from the fact that the stack of curves does not have affine diagonal. This suggests that one needs to consider the stack of curves with an ample line bundle. It is still not clear if the stack of all curves (maybe Gorenstein, maybe reduced) equipped with an ample line bundle is Θ-reductive and S-complete.

A second group studied the question of whether unstable polarized curves have Harder-Narasimhan filtrations. Joshua Jackson explained to the group some computations that he had done along these lines from the perspective of GIT. Since the workshop, this group has shared an overleaf document and is still working on the problem. It seems plausible that in some cases Josh's computations will lead to canonical filtrations of unstable curves.

Is "every" stability condition an instance of Θ-stability?.

One working group discussed how to formulate examples of "semistability" conditions in some commonly studied moduli problems in terms of Θ-stability.

Enlargements of $M_g(\mathcal{X}$.
) for various targets $\mathcal{X}$

Many moduli problems of interest arise as the moduli of (families of) maps from a smooth curve to a target stack $\mathcal{X}$, and it is natural from the perspective of Gromov-Witten theory to look for natural extensions of this moduli problem over the boundary of the stack of Deligne-Mumford stable curves. For example, many known compactifications of the universal Picard scheme arise this way, where the target is $B\mathcal{G}_m$. A lot of work has been done when the target is BG for more general reductive G. The question is whether the existing notions of semistability can be understood as Θ-stability on some larger stack, and whether that larger stack admits a Θ-stratification generalizing the Harder-Narasimhan stratification on the moduli of G-bundles on a curve.

Initial progress: One group worked on the question of whether the stack of nodal curves with a line bundle is Θ-reductive and S-complete, with the goal of understanding Caporoso's stability condition in terms of Θ-stability on this stack. It was observed that one should probably require the line bundle to be ample, and in this case there is a potential method for showing that the stack is Θ-reductive and S-complete. In fact the work in this group lead to the counterexample mentioned in the moduli of curves project. Since the workshop, this group has shared an overleaf document, and is still working on the problem.

Questions around projectivity.

The current state of the machinery typically only produces a proper algebraic space as the good moduli space of semistable objects. When your stability condition is defined with respect to a line bundle, what criteria for projectivity are there? One group reviewed existing approaches to projectivity and discussed several approaches toward a general projectivity criterion for a stack with a good moduli space.

Principal G-sheaves.

The effort to extend the standard results for the moduli of G bundles on a curve to higher dimensional varieties has led to the theory of "principal G-sheaves" and "singular principal G" bundles. What is the natural stack in which these objects (including the non-semistable ones) live? Is there a Harder-Narasimhan stratification of this stack?

Initial progress: One group met every day of the workshop to discuss this problem. It has since resulted in a paper completely answering this question.

Moduli of polarized Calabi-Yau varieties.

There has been tremendous progress in constructing compact moduli spaces of higher dimensional varieties. When the polarization comes from a multiple of the canonical divisor, the general construction of compact moduli spaces was completed as the KSB moduli theory for canonically polarized varieties and the K-moduli theory for Fano varieties. Thus it is a natural question to study compactifications of moduli spaces of polarized Calabi-Yau varieties.

After fixing certain numerical invariants (dimension, volume, etc.) we may assume that polarized klt Calabi-Yau varieties are bounded. From the KSB moduli theory, we know a natural candidate of the class parametrized by the compactified moduli space is (X, L) where X is an n-dimensional semi-log canonical (slc) Calabi-Yau variety, i.e. K_X is torsion, and L is a $\mathbb{Q}$-line bundle. There are several steps in the construction of the moduli stack and moduli space of this class.

(1) Give a proper definition of the moduli functor of $\mathbb{Q}$-polarized slc Calabi-Yau varieties, which yields an algebraic stack $\mathcal{M}$.
(2) Show that $\mathcal{M}$ satisfies the existence part of valuative criterion for properness.
(3) Show that $\mathcal{M}$ is S-complete and Θ-reductive.
(4) Show that there exists a proper algebraic space M parametrizing S-equivalence classes, i.e. a "good" moduli space. The main challenge here is that $\mathcal{M}$ is not of finite type in general, but only locally of finite type, so there might not be a closed point in some S-equivalence class, and the work of Alper-Halpern-Leistner-Heinloth cannot be directly applied.
(5) Show that the Hodge $\mathbb{Q}$-line bundle on $\mathcal{M}$ descends to an ample $\mathbb{Q}$-line bundle on M.

It is interesting to work out this construction in concrete examples, such as elliptic curves, low degree K3 surfaces, abelian varieties, etc.

Initial progress: One group met every day of the workshop to discuss this problem which mainly focused on step 1. They started from moduli functor of polarized elliptic curves, and observed that the limiting curve is a cycle of rational curves, and the $\mathbb{Q}$-polarization corresponds to certain orbi-bundle on the limiting curve with complementary fractional coefficients *at a node. This suggests that one should consider $\mathbb{Q}$-polarizations as Seifert $\mathbb{G}_m$-bundles, and also require that the family of underlying varieties satisfies Kollár's condition. This is similar to a construction by Abramovich-Hassett. Another way to construct the moduli functor is to consider families of log Calabi-Yau pairs $(C_p(X,L), X_\infty)$ of one dimension higher by taking projective cones that satisfy Kollár's condition. This group also discussed the relation between these two moduli functors, and their relation to other moduli problems such as KSB moduli and K-moduli. Since the workshop, this group has made progress on steps 2 and 3, and is still working on other steps of the problem.*

Concrete examples: K-moduli construction.

While the general theory of constructing K-moduli spaces for (log) Fano varieties is completed, people have been trying to work out concrete examples and relate to other well-studied moduli spaces. One group discussed the strategy of describing explicit wall crossing of log Fano K-moduli spaces when varying the coefficients, as the general framework was established by Ascher-DeVleming-Liu. Two notable examples were discussed in this group: (1) Del Pezzo surfaces with bi-anti-canonical sections where one can connect to moduli of K3 surfaces by taking double covers, or moduli of curves by forgetful map; (2) $\mathbb{P}(1,1,2)$ with degree 10 curves where one can relate to KSB-moduli of I-surfaces by taking double covers. Since the workshop, this group has shared an overleaf document, and is still working on the second problem.

Non-reductive GIT.

David Rydh has thoughts on a replacement for the notion of good moduli space that might be useful here, and he spoke about it in the workshop. A working group met to discuss these ideas and generally how one can formulate the results of non-reductive GIT in the context of algebraic stacks.