Noncommutative surfaces and Artin's conjecture

organized by
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Workshop Summary

The central aim of this workshop was to make progress on Artin's conjecture, which is the principal open problem in noncommutative geometry, and plays the role of a noncommutative counterpart of the classical Enriques–Castelnuovo birational classification of projective surfaces. The focus of the conjecture was then divided into four interconnected subareas:

(1) techniques for producing new, possible counter-examples to the conjecture;
(2) obtaining a completed version of Artin's conjecture via a noncommutative Cohen structure theorem;
(3) obtaining a version of Artin's conjecture in positive characteristic;
(4) obtaining a "lifting" result, which would allow one to deduce a version of Artin's conjecture in characteristic zero by knowing positive characteristic analogues.

The participants in the workshop had diverse areas of specialization, including people working in Brauer groups, algebraic geometry, noncommutative ring theory, noncommutative invariant theory, and other fields that fall under the general umbrella of algebra. Consequently, one chief objective of the workshop was to have introductory talks pertaining to the above areas of focus. These were done on the first two days, with an introductory talk on the conjecture itself by Sue Sierra, and talks by each of the organizers covering the following topics: the noncommutative Cohen structure theorem (Satriano), work of Agata Smoktunowicz on Artin's conjecture in positive characteristic (Bell), and a talk about the philosophy of lifting from positive characteristic to characteristic zero as well as deformation problems (Ingalls). Additionally, Paul Smith gave a talk on four-dimensional Sklyanin algebras, which are believed to possibly have homomorphic images that could yield counterexamples to Artin's original conjecture; James Zhang gave a talk on different notions of noncommutative transcendence degree, which play an important role in the formulation of the conjecture.

An open problem session was held on Monday afternoon and several important related questions were raised. For the remaining afternoons, the participants worked on problems motivated by the above four areas of focus. We now give a summary of work done on the above problems.

Potential counter-examples. This impetus for this problem came from a question raised by Toby Stafford during the open problem session on Monday. He made the observation that if one takes a particular four-dimensional Sklyanin algebra, then there is a flat family of prime homomorphic images (parametrized by the complex projective line), each of which provably does not satisfy a polynomial identity and does not possess a non-trivial rank-one valuation. In particular, this observation implies that if, among these homomorphic images, at least one is a domain, then this would yield a counterexample to Artin's conjecture. The group spent several days looking at various members of this flat family, and in each case there were nonzero elements whose product is zero. An important result due to Schmidt and van den Dries asserts that if one has a commutative algebra over a field with d generators and m relations of degree less than D then there is a constant $N = N(d, m, D)$ such that if the algebra is not a domain then there are nonzero elements a and b of degree at most N in the generators whose product is zero. Although no noncommutative analogue of this result exists, this guided the group's approach, as one can search for zero divisors of low degree using a computer algebra package. In each case the group considered, they were able to exhibit an example of nonzero elements of low degree whose product is equal to zero. Ultimately, they were able to show these potential counterexamples did not yield actual counterexamples, giving further underpinning to Artin's conjecture.

Noncommutative Cohen structure theorem. A second group investigated a noncommutative analogue of Cohen's structure theorem. In particular, in this setting one has a complex division ring of transcendence degree two and one assumes that it has a nontrivial discrete valuation ν. (The existence of such a valuation is implied by Artin's conjecture, so one often works with this additional hypothesis.) In this case, one has a valuation subring $\mathcal{O}_\nu$ and a maximal ideal $\mathcal{M}$. One wishes to show that transcendence degree two implies that the division ring $\Delta := \mathcal{O}_\nu/\mathcal{M}$ is in fact a finitely generated field of transcendence degree one. This was achieved during the workshop, using combinatorial results of Smoktunowicz along with Tsen's theorem and work of Small, Stafford, and Warfield. Following Cohen's original approach to proving the structure theorem, one wishes to obtain a description of the completed valuation ring as a type of skew power series ring with coefficients in the field Δ. This typically has two parts: showing that Δ embeds in the completed valuation ring, and then looking at how a uniformizing parameter interacts with Δ. The first part was achieved during the workshop by considering Hochschild cohomology; the second part was also achieved. One thing that came out was that the skew power series structure was potentially complicated and in some cases required one to work with Hasse–Schmidt derivations, which is a family δ_n of higher derivations of Δ, and so one must understand infinitely many self-maps of Δ to understand the structure of the completion. Thus this question was answered during the workshop, but it was realized that the description of the skew power series rings one can obtain was more complex than initially hoped. One remaining question from these lines of investigation is to explain why many of these skew power series structures are not "geometric"; that is, to explain why they do not arise as a completion of a division ring of transcendence degree two. There are currently no plans on the part of this group to consider this remaining question.

Artin's conjecture in positive characteristic. One of the interesting consequences of Artin's conjecture is that it asserts that division rings of transcendence degree two over $\bar{\mathbb{F}}_p$ should be finite-dimensional over their centres. Agata Smoktunowicz proved that this is indeed the case if one has an additional combinatorial hypothesis. This combinatorial hypothesis, while known to hold for many classes of division rings, has the shortcoming of being difficult to verify in general. The third group investigated a possible weakening of this hypothesis, with a view towards proving division rings of transcendence degree two over finite fields are necessarily finite-dimensional over their centres. In this setting, one again assumes the existence of a non-trivial discrete valuation ν, which is used as a proxy for the additional combinatorial assumptions that Smoktunowicz makes. The third group was able to show partial results in this direction, including the case that the division ring has a subalgebra A that is not commutative and has the property that $\nu(A) \subseteq \{0, -1, -2, \ldots\}$. Beyond this, however, obstacles arose that could not be overcome. The third group intends to continue to work on this problem, because it is believed that not all avenues of investigation have been exhausted.

Lifting results and deformation results. The fourth group arguably had the most success of the groups and they believe that their work will ultimately result in a paper that will play an important role in subsequent investigations into Artin's conjecture. During the week this group combined techniques of van den Bergh and recent results on stacks by Geraschenko and Satriano to show that if A is a normal order of global dimension two on a surface Z over a field k such that the rank of A is prime to the characteristic of k, then there is a smooth Deligne–Mumford stack $\mathcal{Z}$ over k with coarse space $\pi : \mathcal{Z} \to Z$, and an Azumaya algebra over $\mathcal{Z}$ satisfying $\pi_* \mathcal{A} = A$, inducing an equivalence of categories $\mathrm{Coh}(\mathcal{A}) \simeq \mathrm{Coh}(A)$. This result has important implications in the study of noncommutative surfaces. In particular, there are questions about whether Artin's proposed list of noncommutative surfaces is stable under the process of taking deformations of a commutative surface. The above result takes an important step in this direction in that it allows one to replace certain deformation problems for singular objects with analogous problems for smooth objects, where significantly more is known thanks to work from Ingalls's thesis.