Carleson theorems and multilinear operators

organized by
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Workshop Summary

The focus of this workshop was on a range of multlinear problems in harmonic analysis. Among them are multilinear Kakeya problems, which is a rapidly growing area at the interface of harmonic analysis, additive combinatorics, geometric measure theory, incidence geometry and algebraic topology; and problems on multilinear singular integrals, which requires very delicate analysis using wave packets on the phase plane. The tools for analyzing these multilinear singular integrals were known to have important applications in a range of other problems, such as the analysis of Carleson's operator, or the Hilbert transform along vector fields; over the years these has developed into a field now known as time-frequency analysis, and problems from this area has also been a central topic of our discussion at this workshop.

The workshop began with two talks on Monday morning. Christoph Thiele gave an introductory lecture on the Hilbert transform along vector fields, bringing us up-to-date to the current status of our understanding of the subject. Michael Christ then talked about extremizers of Hausdorff-Young inequality for sets, and its relation to higher order analysis, which has some potential connections to the analysis of some Carleson-type operators.

On Monday afternoon we had a long problem session. A list of very interesting open questions were collected from an enthusiastic group of participants of the workshop.

On Tuesday morning, Michael Lacey gave a lecture on the bilinear Hilbert transform, and explained its connection to ergodic theory. Then Jonathan Bennett discussed a powerful technique called induction on scales, which is important in the analysis of many multilinear problems; in particular, he explained how this technique can be used to establish the non-endpoint case of the multilinear Kakeya problem.

On Tuesday afternoon we broke into groups, and had many fruitful discussions. We will summarize some of these at the end of the report.

On Wednesday morning Nets Katz gave a lecture on a particular strategy called "stickiness", with which the full Kakeya problem may be attacked. Anthony Carbery then continued with a talk on the multilinear Kakeya problem, this time focusing more on the endpoint case, highlighting its connection to the Borsuk-Ulam theorem in algebraic topology.

On Wednesday afternoon we continued our discussion in problem sessions.

There was only one talk on Thursday morning, by Camil Muscalu on higher order singular integrals. He highlighted the unboundedness of some n-linear singular integrals, revealing the necessity to exploit certain cancellation if some higher order singular integrals were to be bounded. The remaining time in the morning was devoted to group reports, where various groups explain their achievements so far.

Since some groups have already declared victory on their problems, on Thursday afternoon we had another short problem session, where we break into slightly different groups. The discussion then continued in these groups, which also took up the whole Friday morning.

Finally, on Friday afternoon, we had two short talks by some younger participants, where they report on their recent breakthroughs. Vjekoslav Kovac talked about some half-twisted paraproducts in 2-dimensions, and Marina Iliopoulou discussed the counting of joints and multijoints using the polynomial method. The workshop was then concluded with a final round of group reports, where each group explained their findings.

Below we record some of the activities that happened in these groups.

A group worked on the k-linear Kakeya in n-dimensions. They discussed various obstructions, and found an interesting obstruction to using only the Wolff axioms when $k = 3$, $n = 4$.

A group worked on the maximal function along vector fields. They succeeded in bounding the square function of the directional maximal function independent of the number of directions, when the set of directions involved is any order lacunary.

A group worked on multilinear singular integrals, and proved the boundedness of a new $SL(2, \mathbb{R})$ invariant bilinear singular integral, first proposed by Brian Street.

Another group worked on a discrete version of a theorem of Stein and Wainger. There various Diophantine issues arise, and ways were sought to approach these new obstacles.

Yet another group worked on the bilinear circular maximal function. They discussed ways to obtain extra cancellation in the bilinear setting, when an additional rotation is involved.

Finally, a group worked on the Hilbert transform along vector fields. They worked out the dependence of constants on the vector field when the vector field is real analytic, and discussed its implication for Carleson operator.