# Mori program for Brauer pairs in dimension three

organized by
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Workshop Summary

This workshop brought together experts from higher dimensional algebraic geometry and non-commutative algebraic geometry with the goal of developing a version of the Mori program for *Brauer pairs* in dimension 3. Previous work of D. Chan and Ingalls established an analogous program in dimension 2. A successful version of the Mori program for Brauer pairs provides a birational classification of orders, certain gerbes and Brauer-Severi schemes.

A Brauer pair (in dimension $n$) $(X, \alpha)$ consists of a normal $\mathbb{Q}$-factorial quasi-projective variety of dimension $n$ over an algebraically closed field $k$ of characteristic zero where $K_X$ is $\mathbb{Q}$-Cartier and $\alpha \in \operatorname{Br} k(X)$. To develop a Mori program for Brauer pairs in dimension 3, we mimic the commutative situation; we first identify a class of terminal models, then show that any Brauer pair admits a resolution to a terminal model, and that terminal Brauer pairs are stable under Mori contractions and flips.

The main obstruction to this plan in dimension 3 is that we lack a good notion of stable ramification data that would give a cheap class of pairs which are terminal. In dimension 2, the terminal models of Brauer pairs are described by a simple local criterion on the ramification data associated to the Brauer class. The situation appears to be more complicated in dimensions 3 and higher.

## 0.1 Talks

It was necessary to bring non-commutative algebraists up to speed on higher dimensional Mori theory, and conversely, algebraic geometers up to date on recent non-commutative Mori theory in dimension two. As a result, we had 5 introductory talks over the first day and a half. They were as follows:

- **Commutative Mori program** (Kovács): talked about the cone/contraction theorem, the need to introduce mild singularities, classes of which were defined via the notion of discrepancy, the proof that contractions & flips preserve the class of terminal singularities.
- **Mori program for Brauer pairs in dim 2** (D. Chan): talked about results concerning Brauer pairs $(X, \alpha)$ where $X$ is a surface. Geometric invariants called ramification data were recalled, stable ramification data introduced as analogues of smooth varieties, Brauer discrepancy introduced and with it the notion of terminal Brauer pairs, the contraction theorem in this setting was discussed.
- **The coniveau spectral sequence** (Van den Bergh): talked about restrictions on possible ramification data using the coniveau spectral sequence approach.

- **Mori program for 2-torsion Brauer pairs** (Nanayakkara): talked about his recent work on the existence of terminal resolutions for pairs $(X, \alpha)$ where $\alpha$ is 2-torsion and $X$ is arbitrary.
- **Terminal resolutions for toric Brauer pairs** (Ingalls): talked about his recent result with Nanayakkara namely, given a pair $(X, \alpha)$ which is toric in the sense that $X$ is a toric variety and the ramification data of $\alpha$ lies on toric divisors, then there is a projective birational morphism $Y \longrightarrow X$ such that the Brauer discrepancy $b(E, Y, \alpha)$ of any toric exceptional divisor $E$ over $Y$ is positive.

## 0.2 Outcomes

Over the course of the week, we greatly increased our understanding of the Mori program for Brauer pairs in dimensions 3 and higher. The work of Nanayakkara, Ingalls-Nanayakkara attacks the existence of terminal resolutions problem from a completely new direction in quite a subtle way. A picture of the theory is slowly crystallising which reveals fascinating new unexpected phenomena. Our discussions yielded the following:

- The Ingalls-Nanayakkara result has been extended to give actual terminal resolutions for toric Brauer pairs in all dimensions. By using toroidal geometry, we all agreed that this result should in turn give the general existence theorem for arbitrary Brauer pairs.
- Using a variant of the Negativity Lemma [**?**, 3.39], we showed that contractions and flips do preserve the class of terminal Brauer pairs. What was curious about the proof, is that it used so little of ramification theory that it would still work if we modified the definition of discrepancy in certain ways.
- These above results use very little information about the Brauer class. It was noticed that given a function that assigns numbers to all divisors allows us to form a variation of the minimal model program. In the above application this function simply assigns ramification indices of Brauer classes. It could also be ramification of a generic cyclic cover $H^1(k(X), \mu_\ell)$. This would recover the usual minimal model program for the cover.
- We studied all toric pairs of form $(\mathbb{P}^3, \alpha)$ and noticed that the terminal condition cannot be detected locally. This prompted us to look at the question of finding a modification of the definition of discrepancy which would make terminal a local quality. Several suggestions were given and this needs to be investigated further. The robustness of the proof of "contractions/flips preserves terminal" means that a change in definitions should not present problems.
- Although existence of terminal resolutions seems now in sight, the concept of stable ramification data would still be useful and greatly improve our understanding of the Mori program for Brauer pairs in higher dimensions. We approached the problem in two directions. Firstly, we thought about an axiomatic treatment of the class of stable ramification data, and came to the conclusion that the key point is that it need only be stable under blowups at strata. The other approach was to see how we could "improve" the class of log smooth Brauer pairs by blowing up. An exact analogue of the dimension two condition was discovered, but it seems yet more conditions are necessary in higher dimensions.

- We studied the question of stable ramification data in the toric setting. In particular, many examples of terminal Brauer pairs which remain terminal under blowups at strata were given.

Other questions were also raised, but we did not have time to discuss them at length. They include: local models for maximal orders representing Brauer pairs, Brauer versions of the Sarkisov program, comparing the discrepancies of $(X, \alpha)$ and $(X, \alpha^i)$ etc.

## 0.3 Future Plans

From the discussion in the workshop, we believe that at least one paper should eventuate, albeit with quite some work yet to do. Our intention is for an 11 author publication. As a first step, our observations and results have been assigned to various participants, to be typed up. We have now also set up a wordpress blog page to keep in contact with each other. There has already been lively discussion there continuing our workshop group work.

**Bibliography**

[KM98] J. Kollár and S. Mori: *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR1658959 (2000b:14018)