The complex Monge-Ampere equation

organized by
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Workshop Summary

Introduction

Our workshop started with a few words from E. Basor, who introduced the participants (as well as the organizers) to the *modus operandi* of the AIM, including a few precisions about the desired structure of the week to follow.

Very roughly, the meeting pursued two lines of research as follows.

(1) *Comparison between results belonging to real and complex Monge-Ampère theory, respectively.* The elegant theory of real MA operators has been an important source of inspiration for many results in complex setting. Several talks and open problems proposed at the workshop (cf. S. Kołodziej, Z. Błocki, S. Dinew...) have nicely illustrated this.

(2) *Interactions with other fields.* The complex MA operator plays a prominent role in the study of compact Kähler manifolds for which the canonical class has some positivity/negativity properties. This was discussed during the lectures by H. Guenancia, M. Păun. It is equally crucial for the study of the uniqueness properties of canonical metrics (extremal, Kähler-Einstein) via the existence and regularity result for geodesics due to X.X. Chen and Z. Błocki. A lecture pointing towards some of the results and open problems in this direction was given by V. Tosatti, and further questions related to weak geodesics were explored in the talk by E. Di Nezza.

Lectures and Problems

We will comment next some of the lectures/problems which were proposed during the workshop.

• The morning lectures of the first day were given by S. Kołodziej and V. Tosatti, respectively. The main topic of the former talk concerned the integrability exponents of psh functions. Sławomir presented a result in this direction which represents a strengthened version of the inequality

$$\int_\Omega e^{-2u} dV \leq \left(\pi^n + \frac{c(n)t}{(n-t)^n}\right) \operatorname{diam}_\Omega^{2n} \tag{0.0.1}$$

due to J.-P. Demailly, where $\Omega \subset \mathbb{C}^n$ is a hyperconvex domain, and u is psh, zero on the boundary of Ω such that $\int_\Omega (dd^c u)^n = t$.

S.Kołodziej recalled an interesting open problem in this framework: *show that the following inequality holds*

$$\int_{\{u<-s\}} dV \leq c(n) \operatorname{diam}_\Omega^{2n} e^{-2ns/t} \tag{0.0.2}$$

for any s. This would be important for the sharp Moser-Trudinger inequality.

• One of the very interesting problems proposed by V. Tosatti in his lecture concerned the positivity properties of geodesics between two Kähler metrics in the same cohomology class. Let X be a compact Kähler manifold, and let A be an annulus in $\mathbb{C}$. We consider the Kähler metrics $\omega_1, \omega_2 \in \{\omega\}$ and the following homogeneous MA equation on $X \times A$

$$(\pi^\star\omega + dd^c\varphi)^{n+1} = 0 \tag{0.0.3}$$

with ω_j above as boundary condition; here $\pi : X \times A \to X$ is the projection on the first factor.

It is well-known that the best regularity one can expect from φ is $\mathcal{C}^{1,1}$; the main question is to analyze the *strict positivity in the sense of currents* of the restriction

$$\pi^\star\omega + dd^c\varphi|_{X\times\{z\}} \tag{0.0.4}$$

for $z \in A$. An affirmative answer to this question would be very valuable in most of questions involving geodesics.

In connection to this theme, an important problem was proposed by E. DiNezza: *assume that ω_j are only semi-positive, and strictly positive at some point of X. Does the regularity result of Chen still holds?*

• Given a pseudoconvex domain $\Omega \subset \mathbb{C}^n$, the corresponding Bergman kernel verifies the inequality

$$K_\Omega(w) \geq \frac{e^{2nt}}{\{G_\Omega(\cdot, w) < t\}} \tag{0.0.5}$$

where G_Ω is the pluricomplex Green function associated to Ω, and t is any negative number. In order to have a more "explicit" quantity on the left hand side of (0.0.5), it would be desirable that the function $t \to \frac{e^{2nt}}{\{G_\Omega(\cdot, w) < t\}}$ is monotone. Such a result was discussed by Z. Błocki in his lecture; the main tools in the proof (symmetrization of the Laplace operator) were inspired by a crucial result due to Talenti in the context of real MA equation. Z. Błocki pointed out that a complex analogue of Talenti result would give an alternative proof of the crucial $\mathcal{C}^0$ estimate of S. Kołodziej. Other than that, the inequality (0.0.5) can be seen as a local analogue of the famous *partial $\mathbb{C}^0$ estimate*, so further investigations in this direction could be very rewarding.

• The main goal of the lecture by M. Păun was to recall the following conjecture proposed by Y. -T. Siu: *Given a Kähler family $p : \mathcal{X} \to \Delta$ over the unit disk, show that any pluricanonical form on the central fiber of p extends to $\mathcal{X}$*. The projective version of this conjecture was established by Siu in 2002, and it played a crucial role in recent developments of birational algebraic geometry. Concerning the Kähler case, nothing is known in general. An approach via MA operators was suggested, and it is under active investigations. In a similar framework, H. Guenancia presented the state of art in the theory of Kähler metrics with conic singularities, i.e. the main results and techniques in the analysis of the MA equation corresponding to these singular metrics.

• A bit far from the main theme of the workshop, but nevertheless, very interesting was the lecture given by T. Collins. He discussed the so-called ACC conjecture (now a theorem of Ein-de Fernex-Mustaţă): *the family of log-canonical thresholds corresponding to ideals $I \subset \mathcal{O}_{\mathbb{C}^n,0}$ satisfied the ascending chain condition.* The main steps and subtleties in the

original proof of this result were clearly presented. Tristan also formulated a problem which would lead to a new proof, without involving the technique of taking "generic limits of sequence of ideals", which does not have an analytic counterpart (so far...).

Group Sessions

The exchange of ideas during the afternoon group discussions of some of the problems of the list were particularly enjoyable.

The first group read in detail a recent counterexample of Ross-Witt Nyström on the regularity of weak geodesics in the space of Kähler metrics, with the goal of trying to understand whether it could give a counterexample to the strict positivity question mentioned in Tosatti's talk. It was quickly realized that this is not the case, since the counterexample is on the product of the manifold and a disc (as opposed to an annulus), and then the rest of the meetings were spent analyzing the precise failure of regularity of the counterexample, which is itself a very delicate issue.

The second group looked at the problem of defining the Monge-Ampère operator for certain unbounded quasi-plurisubharmonic functions so that the resulting measure charges analytic sets. The case of limits of the continuity method on a Fano manifold which is not Kähler-Einstein was studied in detail, and some progress was made towards showing that the limiting quasi-psh function should agree with a Green-type function on the manifold with singularities along a suitable multiplier ideal subscheme.

The third group studied the problem of obtaining (or disproving) an interior Pogorelov-type C^2 estimate for the complex Monge-Ampère equation. A rough strategy was discussed towards proving this estimate by "integrating" the zero directions of the complex Hessian to produce a harmonic disc, but extreme technical difficulties seem to prevent this strategy from being brought to fruition.

The fourth group studied Păun's question related to invariance of plurigenera. An ambitious goal would be to show that the solution of a family of fiberwise twisted Kähler-Einstein metrics, which is known to be globally nonnegatively curved, has minimal singularities. It was observed that this is the case when the relative canonical bundle is nef and big, but not much more was said in general.

The last group studied Guenancia's question of limits of twisted conical KE metrics when the twisting parameter goes to zero, and made some promising initial observations.