Moments in families of L-functions over function fields

organized by
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Workshop Summary

Introduction

This workshop was devoted to studying interactions between homological stability and asymptotic questions in number theory over function fields. In recent years, homological stability of suitable Hurwitz spaces has been used to make great progress on function field cases of Cohen-Lenstra heuristics (Ellenberg-Venkatesh-Westerland and Landesman-Levy), Malle's conjecture (Ellenberg-Tran-Westerland and Landesman-Levy), the Conrey-Farmer-Keating-Rubinstein-Snaith predictions for moments of L–functions (Bergström-Diaconu-Petersen-Westerland, Miller-Patzt-Petersen-Randal-Williams), and heuristics on Selmer ranks (Ellenberg-Landesman). The workshop brought together researchers both from analytic number theory and topology, and allowed two groups with different backgrounds to work together on problems and further explore the connections between the two fields.

The intent of the organizers was to bring the new developments in homological stability to a wider audience, and to foster sharing of ideas for current problems with an arithmetic flavor using topological tools. Feedback from the participants was overwhelmingly positive, both from the number theorists and the topologists. We made an effort to have equal amounts of speakers from both communities for the morning lectures.

Working Group Summaries

Out of the proposed problems during the problem session, the organizers picked nine. By the end of the week, there were five problems on which the different groups were actively working. Here, we summarize the five problems that were studied during the afternoon sessions.

Image of the Burau representation specialized to roots of unity.

This group investigated the questions of determining the image of the Burau representation specialized to roots of unity ρ. For $\rho = -1$, this is known by the work of A'Campo and Bloomquist-Patzt-Scherich. This was a key ingredient in recent progress on Conrey-Farmer-Keating-Rubinstein-Snaith heuristics for moments of families of quadratic L–functions over function fields by Bergström-Diaconu-Petersen-Westerland and Miller-Patzt-Petersen-Randal-Williams. Knowing the image of the Burau representation specialized to other roots of unity should play a similar role in studying families of higher degree L–functions. For $\rho = -1$, the image is a certain congruence subgroup of a symplectic group. For larger roots of unity, it was proved during the workshop that the image of the specialized Burau representation is contained in a certain congruence subgroup of a unitary group over $\mathbb{Z}[w]$, where w is a primitive cubic root of unity. The group hopes to show the image is in fact

this congruence subgroup. The participants in the group also found a simpler proof of the results of A'Campo and Bloomquist–Patzt–Scherich (the case $\rho = -1$) and are working on adapting this approach to the case of primitive third roots of unity. The fact that the level 3-modular curve has genus zero and the fact that the Eisenstein integers are a Euclidean domain seem relevant. The group also considered approaches bases on Armstrong's method for finding generators of a group via group actions on highly connected simplicial complexes with highly connected quotients.

Distribution of epsilon factors

This group worked on the following problem. Fix a finite group G and a nontrivial irreducible representation ρ of G. Motivated by classical and modern work in analytic number theory on the distribution of Gauss sums, this group studied a function field analog; namely, the average of the epsilon factor of ρ for a random G–cover of a curve (focusing on the line). Methods of analytic number theory, such as multiple Dirichlet series, are applicable in this case as well and were discussed in detail.

An approach based on work of Sawin-Lipnowski-Tsimerman and on homological stability results of Landesman-Levy for racks was suggested. For a variant of the problem when G is cyclic, a strategy based on homological stability for commutators of (pure) braid groups is being developed by Miller–Tosteson.

Trigonal curves

It has been shown that the number of cubic field extensions of $\mathbb{Q}$ of discriminant at most X is $C_1X + C_2X^{\frac{5}{6}} + o(X^{\frac{5}{6}})$, for explicit constants C_1 and C_2. A similar result has been shown over $\mathbb{F}_q(t)$. These results have also been proven when one restricts to such covers of squarefree discriminants, also known as simply branched covers. The presence of the secondary term $X^{\frac{5}{6}}$ suggests that the Hurwitz space of degree 3 simply branched covers may exhibit secondary stability, as discussed in recent papers by Galatius, Kupers, and Randal-Willliams.

The goal of this group was to see if one could understand the optimal stable range of this cohomology and perhaps prove secondary stability.

The key idea the group used to get started is that there is a spectral sequence relating the cohomology of the Hurwitz space to the cohomology of a related Nichols algebra. Using this idea, the group was able to compute all the unstable first homology of the Hurwitz space of simply branched covers, and give geometric explanations for it in terms of divisors. The group thinks there should be a particular class in $H_2(\mathrm{Hur}_6)$ which accounts for secondary stability with slope 1/3, though perhaps this class only exists after quotienting by the stabilization isomorphism. Geometrically, this class should be related to the class of trigonal curves living on extreme Hirzebruch surfaces, as was previously observed in the function field counting problem over $\mathbb{F}_q$. The group is still working on identifying a specific candidate class which would give the secondary stability.

Negative moments and ratios of L–functions

This group studied several questions related to negative moments of L–functions over function fields and over number fields.

In one direction, the group tried to understand if the homological stability techniques used for the study of positive moments in the work of Bergström-Diaconu-Petersen-Westerland and Miller-Patzt-Petersen-Randal-Williams can be used to obtain formulas for the negative moments of L–functions over function fields when the shifts in the denominator go to 0 with the size of the family. Previously, these techniques were successfully used by V. Wang to prove the Ratios Conjecture when the shifts in the denominator are at least a fixed distance away from the critical line. It was deemed that Wang's argument in the study of the Ratios Conjecture with bigger shifts is essentially optimal and does not allow one to let the shifts go to 0. In a different direction, the group also sketched an argument, using previously proven formulas for the negative moments with small shifts, which would allow one to obtain lower bounds on the unstable homology in the case of quadratic L–functions over function fields.

The group also worked on formulating conjectures for the negative moments of L–functions when the shifts are very small (less than $1/(\log \mathcal{C})$, where $\mathcal{C}$ denotes the conductor of the L–functions in the family). Previously, asymptotic formulas for the negative moments were formulated by S. Gonek for $\zeta(s)$; more recent random matrix computations which originated with work of Berry-Keating partially contradict the number theoretic conjectures in certain ranges. The group worked on developing a "hybrid model" for the negative moments, writing the L–function as a product between a partial Euler product and a partial Hadamard product, which would provide a different way of predicting asymptotic formulas in these cases.

Secondary stability

Since the discovery of secondary homological stability by Hepworth and Kupers-Galatius-Randal-Williams in 2016, there has been a folklore expectation that the correspondence between homological stability and main terms in arithmetic counts featuring in works like Ellenberg-Venkatesh-Westerland should admit an enrichment to match secondary stability with secondary terms in arithmetic statistics.

The motivating problem for this working group was to provide the first nontrivial example where secondary stability could account for a secondary term in arithmetic statistics, and to provide a precise mechanism/paradigm for this correspondence that could be applied to known examples.

For the first two days, the group pursued the example of counting abelian extensions of a function field and specifically cyclic 3 extensions of $\mathbb{F}_q(t)$ for $q \equiv 1 \pmod 3$. From class field theory, one can obtain a straightforward count with main and secondary terms. The group considered the sequence of bicolored configuration spaces and its rational homology. A feature/challenge of this example is that there are two stabilization maps and the group tried to understand what stable ranges should mean, how they corresponded to the main term of the arithmetic count, and how one could begin to account for separating main from secondary terms.

The group also worked on constructing an axiomatic/algorithmic framework which would take as input a sequence of spaces with known stability, secondary stability, etc. ranges, as well as computations of the stable, metastable cohomology and sub-exponential bounds on the unstable pieces. The algorithm would then output an assymptotic count of the

corresponding arithmetic problem and with an explicit correspondence between main terms, secondary terms, etc. of this count and the input stability, secondary stability, etc. Drawing inspiration from Ishan Levy's talk on his work with Aaron Landesman, the group sketched an algebraic procedure for iteratively working through the stabilization maps (taking homotopy localizations to get the stable term, then homotopy cofibers to subtract it off) and matching the output to the terms in the polynomial in n coefficient of the leading exponential term. What emerged as a goal is a recursive procedure for going from the k^{th}-stability to the $(k+1)^{\text{st}}$-stability and extracting the $k+1$ term of the count. The "edge terms" at each stage appear to have not received much attention in the homological stability community, though they are very natural from the perspective of arithmetic statistics. It was suggested the topological meaningfulness of the "edge terms" may be a general phenomenon, or at least one that encompasses examples with Hurwitz spaces.