Positivity, graphical models, and modeling of complex multivariate dependencies

organized by

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Workshop Summary

1. Overview

Understanding complex multivariate structure and dependencies in very high dimensions is a fundamental challenge of the 21st century. Problems of this type arise naturally in various domains: modeling gene-gene associations in cancer and cardiovascular disease, detecting climate change, and understanding how financial instruments interact leading up to and during the collapse of financial markets. One of the most fundamental measures of dependencies used in such applications are covariance/correlation matrices. The goal of the workshop was to study the properties of techniques that can be used as alternatives to ℓ_1 methods to obtain sparse estimates of covariance/correlation matrices, as ℓ_1 techniques require solving optimization problems that are often too computationally expensive for modern-day applications. One approach that shows much promise consists of applying functions entrywise to matrices in order to induce sparsity. This method does not rely on convex optimization and is immediately applicable. However, for this technique to be useful in downstream applications, it is important that the resulting matrices be positive definite. In order to understand properties of entrywise mappings, we gathered a community of analysts, linear algebraists, and experts working in the area of graphical models.

The workshop was a great success. Participants coming from different communities and backgrounds interacted together, leading to rich exchanges of ideas. Many of the focus groups made significant progress on their problems, and have continued to work together after the workshop. The workshop has been a tremendous opportunity for many of the participants. Many of them found new collaborators, and got interested in multiple new problems.

2. Highlights of the workshop

The mornings at AIM were filled with talks by Rajaratnam, Guillot, Khare, Hiai, Friedland, Putinar, Lauritzen, and Bhatia. During the first day, Bala Rajaratnam gave the opening talk in order to set the stage for subsequent talks in the workshop. Rajaratnam's talk connected the area of positivity on one hand and graphical models and modeling multivariate dependencies on the other. More specifically, Rajaratnam's opening presentation, which lasted almost an hour and a half, connected two major areas of research in the mathematical sciences, one in pure mathematics, analysis in particular, and the other in theoretical statistics. In doing so, the talk described the focus area of the workshop and the connections between the various disciplines represented by the participants. The talk contained both overview of the subject area as well as technical material. The talk first described the screening of correlation and partial correlation matrices as advocated by the approach of Hero and Rajaratnam in the purely high dimensional setting. The talk then demonstrated

the various benefits of correlation screening, and how correlation screening naturally led to questions of positivity. The various ways in which positivity arises in modern settings was explained carefully. This included technical results on characterizing functions preserving positivity when such functions are only applied to off-diagonal entries of the cone. The second setting involved thresholding positive definite functions and extensions of Pólya's work in this area within the context of modern high dimensional covariance estimation. Rajaratnam's talk was followed by two technical talks by Dominique Guillot and Apoorva Khare about entrywise functions preserving positivity under rank and sparsity constraints respectively. During the second day, Fumio Hiai discussed his recent work on powers preserving positivity, monotonicity, and convexity. This was followed by Shmuel Friedland who spoke about low-rank approximation of large matrices, and about entanglement of quantum systems. On the third day, Mihai Putinar spoke about positive definite functions on spheres, and significant generalizations by Pólya, Quillen, and recent joint work with Scheiderer. Steffen Lauritzen then talked about his recent work on Linear Estimation of Concentration Matrices. On the fourth day, Rajendra Bhatia gave a lecture about the history of a problem he had proposed during the workshop, involving the inertia of a family of matrices.

During the afternoon, participants broke off into several research groups, which worked on problems proposed by the participants. Many of the groups made significant progress on their problems. These activities are now described.

(1) One of the groups focused on the question of characterizing functions preserving positivity for matrices of a fixed dimension. Following results from Schoenberg and Rudin, it is known that an entrywise function f preserves positivity on matrices of all dimensions if and only if f is absolutely monotonic on the positive real axis (i.e., if f admits a Taylor series with nonnegative coefficients). However, the problem of characterizing functions preserving positivity in a *fixed* dimension has been open for more than 60 years. During the workshop, the group made significant progress in understanding how close has a function to be to the class of absolutely monotonic functions when preserving positivity on $n \times n$ matrices. The group already has a write-up of about 15 pages that they expect to publish in the coming months.

(2) A second group made significant progress on a longstanding conjecture by N. Wermuth concerning properties of MTP2 distributions and graphical models. One such property is faithfulness. The group also showed that any independence model generated by an MTP2 distribution is a singleton-transitive compositional semigraphoid. In addition, the group studied the Markov properties. The group was a good example of cross fertilization of ideas. In particular, during the workshop, an expert in linear algebra (Shaun Fallat) worked closely with the graphical models community. Equipped with new insights from linear algebra, the group was able to resolve Dr. Wermuth's conjecture.

(3) A third group looked at problems of variable ordination related to graphical models and covariance estimation. In particular, the group looked at applied mathematics interpretation of a statistical method introduced by Rajaratnam and Salzman for variable ordination called Best Permutation Analysis (BPA). Group members discussed that the variable ordination technique introduced Rajaratnam and Salzman gives a theoretical optimization characterization to certain algorithms in optimization, computational mathematics, numerical analysis, and computer science. In fact, due to BPA, such algorithms can now be viewed as yielding solutions to optimization

of certain functionals on the symmetric group. The group also put in a SQuaREs proposal to continue working on the project.

(4) A fourth group studied a problem raised by F. Zhang concerning functions preserving positivity when applied to block matrices. The group now has a much better understanding of the problem, and the participants expect to continue interacting together in the future.

(5) A fifth group worked on a conjecture raised a few years ago by R. Bhatia concerning the behavior of the eigenvalues of entrywise powers of specific matrices. Progress has been made on the conjecture during the workshop. We are pleased to announce that Professor Bhatia together with the help of other workshop participants have resolved the conjecture shortly after the workshop.

In summary, the workshop was a solid success. It led to rich interactions between communities who would not have interacted otherwise. Many of the proposed problems were solved, and many durable links have been created between researchers who expect to continue working together after the workshop.