Configuration spaces of linkages

organized by
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Workshop Summary

Talks

The workshop followed the usual AIM workshop style, with two talks each morning and informal activities in the afternoon.

Michael Farber kicked off the workshop by giving a talk on the topology of the configuration space of polygonal linkages in 2D. He reviewed the usage of Morse theory to study the manifold structure and Betti numbers of configuration spaces via Morse critical points. These are the collinear configurations for polygonal linkages. He also gave a brief introduction to his proof of Walker's conjecture that the topology of the configuration space determines the linkage up to a permutation orbit of chambers in the space of edge-length-vectors. These are delineated by walls determined by edge-lengths that permit critical or collinear configurations. The chambers are determined by the short subsets i.e. the set of bars of the polygon whose sum of lengths is less than the sum of the lengths of the remaining bars.

Gaiane Panina gave a talk on the cell decomposition of the configuration space of 2D polygonal linkages via the face and incidence structure of permutahedra. The configurations are put into equivalence classes using a cyclic labeling of the points, and each k-cell corresponds to a cyclically ordered partition of $k+3$ non-empty short subsets. Such a cell decomposition directly leads to an efficient motion planning algorithm.

The talk by Ileana Streinu described the usage of pointed pseudo-triangulations to achieve an expansive motion of 2D linkages, i.e. an infinitesimal motion where no pairwise distance decreases.

This led to an efficient algorithm for solving the well-known carpenter's rule problem, which asks whether a polygonal linkage in an arbitrary 2D configuration can be unfolded to a convex configuration using non-crossing motions.

The talk by John Owen discussed the complexity of linkage configurations via the Galois group of frameworks. In particular, a graph is said to be quadratically or radically soluble (QRS) if every point in its configuration lies in a radical extension over the bar lengths.

He gave a class of inductive graph operations which maintain the solubility of Galois groups of the original graph. This was used to show that for planar graphs, a class of linkages called tree-decomposable linkages are equivalent to QRS graphs.

He also gave a recent rigidity characterization of generic 2D point-line framework with line-line angle, point-line distance and point-point distance constraints.

A different perspective describes the configuration space of linkages using Cayley configuration spaces, i.e. the set of all achievable distances on certain non-edges, whose addition as bars to the linkage makes the linkage rigid.

The talk by Menghan Wang was about the 1-dof tree-decomposable linkages introduced by Owen, but additionally with low Cayley complexity, and all interval endpoints in the Cayley configuration space being QRS values. She gave characterizations for such graphs and an efficient motion planning algorithm via the Cayley configuration space.

Meera Sitharam's talk showed that the class of graphs, whose Cayley configuration space on any subset of edges for any d-dimensional linkage is convex, is exactly the class of d-flattenable graphs. For the proof she introduced the idea of viewing different types of rigidity and configuration space problems from the perspective of Euclidean distance matrix cones.

She introduced the software EASAL that efficiently computes the so-called atlas of a molecular or sticky sphere assembly configuration space. The short-ranged potentials and sterics (atoms want to be in contact, but cannot interpenetrate) are geometrized, then a natural Thom-Whitney stratification is computed followed by Cayley convexification of the regions. EASAL not only generates the atlas but also efficiently samples it in various distributions, and computes integrals over configuration space regions of various dimensions.

Miranda Holmes-Cerfon gave a related talk on configurations of identical sticky spheres that form clusters. The geometrization and stratification are identical to EASAL. Cayley Convexification is not employed, rather a direct parametrized sampling of the the Cartesian configuration space is used to find all 0-d (rigid clusters) and compute integrals of 1-d paths and 2-d regions. She talked about reachability of rigid clusters from each other in the network of 1-d paths in the configuration space (each path is a 1-dof structure). In particular, she is interested in calculating the "volume" of the space of singular clusters asymptotically from the potential, as well as calculating the volumes of all of the manifolds in the configuration space, when the particles are coloured and have different strengths of interaction.

The talk by Bernd Schulze was mainly about linkages with forced symmetry. He reviewed several results on symmetry-forced Laman type theorems in 2D for different symmetry groups, including rotation groups. He is also interested in the symmetry-forced rigidity for the dihedral group, as well as linkages with incidental symmetry.

Abdo Alfakih introduced the dual rigidity matrix obtained from Gale transformation of the point configuration. He also gave a different way of describing the set of equivalent frameworks via the projected gram matrix and the set of non-edges. This approach led to a characterization of universal rigidity, i.e. there exists only one equivalent framework in all dimension, via dimension rigidity, i.e. there is no equivalent framework with a higher dimensional affine span.

Problems and working groups

Working Group 1 Based on Farber's talk, Meera Sitharam asked the problem of extending the Morse theory approach to study the critical points, chambers, and Euler characteristic of graphs of tree-width 2.

Panina's talk led to the problem of studying the configuration space of 3D polygonal linkages using the Cayley configuration space and small covers. A working group was formed for these problems with the following members: Michael Farber, Gaiane Panina, Meera

Sitharam, Menghan Wang, Ileana Streinu, Ciprian Borcea, Herman Servatius, Troy Baker, Joel Willoughby. An initial result has been established that shows that for graphs of tree width-two, a configuration is in a Morse-Bott if and only if there exists a collinear simple cycle.

Farber's talk led to the problem of the asymptotic behaviour of C_n (the number of orbits of chambers in the case of polygonal linkages, which turns out to be equal to the number of hyperplane partitions of the vertices of the n-cube). Farber asked the following problem about configuration spaces of closed polygons in $\mathbb{R}^3$: consider connected components U_i of the space of configurations without self-intersections. Are these components contractible? What can we say about their topology? Farber and Panina gave the following problem: consider a closed polygon in $\mathbb{R}^3$. According to Klyatchko, the configuration space M_l has a symplectic form for for volume $Vol(\sqcup U_i)/Vol(M_l)$. The question is to understand this proportion.

Working Group 2 Streinu's talk led to the long open problem of the Carpenter's Rule problem in 3D, which asks whether a open chain in 3D consisting of unit-length bars in an arbitrary configuration can always be unfolded into a straight line configuration using non-crossing motions. A working group formed with the following members: Dmitry Berenson, Maria Hempel, and Joel Willoughby.

An approach of forming the problem into a convex optimization problem was suggested.

Working Group 3 Owen gave the following problem based on his talk: when is the Galois group of a specialized rigid framework a subgroup of the Galois group of a generic framework of the graph?

A working group is formed on this problem with the following members: John Owen, Jessica Sidman.

Working Group 4 Michael Thorpe gave the following problem given a finite network of corner-sharing triangles in the plane, where each cycle of triangles has size 5,6,7,8,9 with an average of 6. Prove that pinning every other boundary vertex (vertex of degree two) gives an isostatic network, and study the case when the triangles are equilateral. A working group is formed with the following members: Michael Thorpe, Louis Theran, Brigitte Servatius, Elissa Ross, Tony Nixon, Meera Sitharam.

A proof was given in the generic and geometric setting and related questions were developed.

Working Groups 5 and 6 Audrey Lee-St. John posed the following problem: given a multi-robot formation which is a generically minimally rigid framework G with diameter $D = max_{(p_i,p_j)}\|p_i - p_j\|$, we remove an edge and obtain $\bar{G}$ with minimum diameter d over its configuration space. Understand the relationship between D and d, and find algorithms to detect which edge to delete for maximal change in diameter. Abdo Alfakih and Shinichi Tanigawa worked on this problem.

St.John gave another problem on persistence theory with symmetry: given multi-robot formation where each node has out-degree 2 except for the leader and co-leader, is there a way to take advantage of symmetry in these graphs in an effort to reduce the amount of information needed to remain persistent? More specifically, instead of nodes/robots in each orbit sensing their corresponding neighbors, is there some (rigorous) way to reduce the sensing done and perhaps make use of "cheaper" communication between orbits? The

working group has the following members: Audrey Lee-St.John, Bernd Schulze, Troy Baker, Shinichi Tanigawa.

NOTE: A webpage is created at http://www.cise.ufl.edu/∼tbaker/aim which contains lecture notes for the talks, as well as problem statements and progress reports from all working groups.

The following problems were additionally discussed.

The following conjecture by Walter Whiteley, which requires further formalization, was mentioned in Schulze's talk: given a symmetric bar-joint framework, does the configuration space (with appropriate parts of the frame of reference fixed) have the symmetry of the most symmetric individual realization in the configuration space?

Schulze asked the question whether the pseudo triangulation algorithm for the Carpenter's Rule give some unfolding that preserves symmetries, which is shown to be false. Note that the original scheme by Connelly, Demaine, Rote does preserve symmetry.

Schulze also gave the following symmetry related problems: (1) Suppose a symmetric framework (linkage) has a 1-dof expansive mechanism. Does the mechanism preserve the symmetry? (2) Under what conditions is a linkage guaranteed to preserve the original symmetry throughout the configuration space?

Maria Hempel gave the following problem: what is the dimension d of the configuration space of genus g triangulated polyhedra with fixed combinatorics, parameterized by dihedral and face angles? It is proved that when $g = 0$, $d = |E| - 1$. The conjecture is that $d = |E| - 1$ holds for general g.

Louis Theran asked the following problem: consider a Delaunay triangulation with fixed combinatorics. For $d = 2$, the configuration space is a ball. What is the configuration space for other d? Is it universal for $d = 3$?