K-stability and related topics

organized by
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Workshop Summary

The focus of this workshop was on reviewing recent progress on K-stability of Fano varieties, investigating connections between various approaches and encouraging participants from different backgrounds to share techniques with each other. In particular, discussed K-stability from the viewpoints of the minimal model program, differential geometry, singularity theory, moduli theory, and non-Archimedean geometry.

The following is a list of important topics related to K-stability that we covered:

(1) K-stability of explicit Fano varieties, especially in dimension three.
(2) The delta-invariant of Fano varieties and valuations computing this invariant.
(3) The normalized volume of klt singularities and the stable degeneration conjecture.
(4) Positivity of the CM line bundle.
(5) Reduced uniform K-stability of Fano varieties.
(6) The non-archimedean approach of K-stability, and the entropic regularization conjecture.

Two conjectures emerged as crucial in the field and were discussed during the workshop:

- The (local and global) Higher Rank Finite Generation conjecture
- Equivalence between reduced uniform K-stability and K-polystability

The workshop was very enjoyable and valuable, and allowed the participants to take stalk of recent progress in the field and summarize what some of the important open problems are. In addition, progress was made on several problems during the problem session, which furthermore sparked exchanges of ideas that will surely lead to significant new research.

1. Talks

The workshop was organized as in the usual AIM format. There were two introductory talks each morning, whereas the afternoons were reserved for working in groups. The talks were as follows.

Chenyang Xu gave a survey on K-stability of Fano varieties from the viewpoint of valuations and filtrations, and explained the importance of the higher rank finite generation conjecture.

Ivan Cheltsov explained the role of alpha-invariant and the delta-invariant of Fano varieties, and how to go about computing the. He also exhibited many classes of Fano 3-folds which are interesting from the viewpoint of K-stability.

Mattias Jonsson explained various non-Archimedean aspects of K-stability for general polarized varieties. He also highlighted the entropic regularization conjecture.

Harold Blum surveyed recent work on the the construction (boundedness, openness, separatedness, etc) of K-moduli spaces of Fano varieties. In particular, explained the recent proof of the openness K-semistablility.

Carolina Araujo explained several interesting Fano manifolds with a great deal of symmetry and explored the theory of higher-Fano manifolds from the viewpoint of families of minimal rational curves.

Chi Li discussed recent work on the the Yau-Tian-Donaldson conjecture and the concept of reduced uniform K-stability for Fano varieties. He also showed how the entropic regularization conjecture implies the YTD conjecture for general polarizations.

Nathan Grieve explained that the S-invariant that appears in the definition of the delta-invariant naturally also features in diophantine geometry.

Yuchen Liu gave a survey on the normalized volume for klt singularities and the stable degeneration conjecture. He also explained the relationship to K-semistability of Fano varieties.

Zsolt Patakfalvi recalled the definition of the CM line bundle for families of Fano varieties, and explained its positivity when the generic fiber of the family is uniformly K-stable.

Ziquan Zhuang explained the Harder-Narashimhan filtration of the pushforward of relative anti-pluricanonical divisors of families of Fano varieties over curves from K-moduli-theoretic viewpoint.

2. Problem sessions

After a general discussion of possible topics during the first afternoon, we settled on four different problems that were then discussed during the remaining afternoons. Here is a summary of the problems discussed.

- **"K-stability of Fano threefolds" (presented by Ivan Cheltsov):** This group considered K-stability of several Fano threefolds. They found an effective approach to show the K-stability of the blowups of del Pezzo threefolds along elliptic curves. Moreover, they showed the strict K-unstability of the blowup of the del Pezzo threefold of degree five along a bad line.
- **"Birkar's theorem for $\alpha > 1$" (presented by Jingjun Han):** This group considered the following problem: If a Fano variety X satisfies $\alpha(X) > 1$, can we compute

$\alpha(X)$ by a single $\mathbb{Q}$-divisor?". (The case when $\alpha(X) \leq 1$ was proved by Birkar himself.) The group reduced this problem to a certain uniformity property of singularities, and obtained several results.

"Special degenerations of $\mathbb{P}^2$" (presented by Ziquan Zhuang): This group tried to classify all special degenerations of the projective plane. There are two steps. In the first step, they attempted to show the finiteness of such special degenerations and gave effective steps to classify them, which is a special case of the Higher Rank Finite Generation conjecture. They also try to show that there are only very small number (4) of possible degenerations.

"General polarization and filtrations" (presented by Mattias Jonsson): Here the idea was to understand K-stability for general polarization, given some spectacular recent progress in the Fano case. In particular, the group discussed the problem of regularizing a non-Archimedean metric of finite energy while controlling the entropy. By the work of Chi Li, this problem is a key missing part to establishing the general YTD conjecture.