# Generalized Kostka Polynomials

organized by
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Workshop Summary

**Goals.** Kostka polynomials and their generalizations have arisen in numerous ways such as in the context of symmetric functions, combinatorics, representation theory, quantum groups and crystal bases, statistical mechanics, algebraic geometry, and Kazhdan-Lusztig theory. There are several open problems concerning them, such as positivity, finding a combinatorial expression, and whether the various generalizations that arise in different contexts coincide. The goal of this workshop was to bring together mathematicians who have studied Kostka polynomials from different points of views, state the various connections and open conjectures, and work towards their proofs. Our secondary goal for the workshop was to foster collaborations across these viewpoints.

**Participants.** To achieve our goals, we brought together experts from the different fields in which Kostka polynomials arise to discuss the most recent results and to compile a list of open problems and conjectures. One person from each major group was asked to present the context in which Kostka polynomials arose in their research and the corresponding definitions, as not every participant was familiar with all of the different generalizations Kostka polynomials have.

This group of mathematicians consisted of combinatorialists, representation theorists, geometers, mathematical physicists, and computationally oriented mathematicians. Among the participants, we also included a number of graduate students, recent graduates, mathematicians from primarily teaching colleges and one from industry seeking to return to academia.

**Structure.** The structure of an ARCC workshop was very well-suited to our objectives and needs. At the AIM staff's advice, we arranged for a small number of hour-long introductory lectures, coupled with several 20-minute talks. This allowed people from different areas to see the other interpretations and definitions of generalized Kostka polynomials. While most of the introductory lectures were given by senior mathematicians, we encouraged the junior mathematicians to give the shorter talks. We also had three moderated discussion sessions where open questions and problems were posed and discussed. This type of session was new for most participants and it took them some time to adjust to the format. However, once a few participants spoke up, others started to contribute problems as well. The organizers were a bit hesitant to have an open problem session the first day, but looking back, strongly agreed that it was a great idea. Finally, we allowed for several large blocks of time for small group meetings. During this time, some small groups started to work on some of the questions raised during the discussion sessions, while others used the time to continue already existing collaborations. In general, the fairly open schedule encouraged many useful discussions and paved the way for new projects and collaborations.

The conference opened with a mixture of introductory lectures and short talks that introduced different constructions of Kostka polynomials, the contexts where they appear, and their representation-theoretic and geometric significance. A few more lectures followed throughout the week. In addition, during the discussion sessions, several questions came to the fore, and were discussed during the small group meetings. There were roughly eight different groups that consisted of from two to ten members, some of the smaller ones continuing collaborations and others starting new collaborations. At the close of the workshop, we had a wrap-up session, where groups presented the progress they made on their problems.

**Talks.** We had four overview lectures placing the generalized Kostka polynomials into the larger contents of where they appear, accompanied by six short lectures designed to give the precise definitions.

In one of the short talks, Nick Loehr defined the (modified) Macdonald and $q, t$-Kostka polynomials, and gave a combinatorial formula of the former. A combinatorial formula of the latter is still an open question of interest, although their positivity is known and implied by a recent result of Haiman that uses the geometry of the Hilbert scheme. Haiman's $n!$ Theorem is a major recent breakthrough in the field. Mark Haiman gave an introductory lecture on his theorem and proof. He also discussed related theorems and conjectures for diagonal harmonics.

Both talks were related to the second short lecture presented by Thomas Lam that defined LLT (ribbon) polynomials. Recent work of Haiman-Haglund-Loehr shows the conjectured Schur-positivity of LLT polynomials would give another proof of the Schur-positivity of Macdonald polynomials (and hence the positivity of the $q, t$-Kostka polynomials). After many requests from participants, we added another short talk by Francois Descouens on the representation-theoretic interpretation of the LLT polynomials. This interpretation involved quantum groups and canonical bases.

Masato Okado presented an introductory talk on Kirillov-Reshetikhin crystals, from the viewpoint of statistical mechanics. Here, the statistic of energy was defined, which is used to define generalized Kostka polynomials for rectangular shapes (and level-restricted generalized Kostka polynomials). Related to this was a short talk by Rinat Kedem on the fusion product and level-restricted fusion product. Open problems on this topic were the main focus of the third problem session. It was also asked whether this could lead to a meaningful definition of two-parameter level-restricted Kostka polynomials.

We opened the second day with a lecture by Jennifer Morse on the history of and recent results concerning $k$-Schur functions. The speaker listed many open problems in her talk, and connections to the open problems that already came up in the earlier session. Questions about the $k$-Littlewood-Richardson coefficients, the relationship between $k$-Schur functions and LLT polynomials, the representation-theoretic interpretation of $k$-Schur functions, and the role of the affine Hecke algebra in this picture were also asked in the subsequent problem session.

John Stembridge presented an introductory lecture on generalized Kostka polynomials (the $K^{\psi}_{\lambda,\mu}(t)$ mentioned below) and Kazhdan-Lusztig polynomials in arbitrary type. The final overview lecture by Arkady Berenstein was on geometric crystals. Other short talks included one by Mike Zabrocki on creation operators, and another by Christoph Schwer on galleries. Galleries are something that lies between tableaux and Littelmann paths in complexity, but work in all types. They also highlight the role the affine Hecke algebra plays in the Hall-Littlewood polynomials. In an open problem session, it was asked whether one could find a charge statistic on galleries. One postdoc who was very interested in Littelmann paths commented that this suggested to him a new avenue of research to pursue.

**Group work.** On the third day, we formed small groups to work on a subset of open problems, and the small groups continued to meet throughout the workshop.

One group of two that was a continuing collaboration worked on the problem of quasi-invariants, which are conjectured to be of dimension $n!$. At the degenerate case ($r = 0$), it covers a classical result on covariants; at the next step ($r = 1$) it is a result of Hivert on quasi-symmetric functions. The participants were searching for a distinguished basis that worked across cases.

Another group was using Kostant's partition function to define a generalized (one parameter) Kostka polynomial in any type, depending on a subset $\psi$ of the weight lattice. This was based on the material presented in one of the introductory talks. They reduced testing for positivity of the linear term to describing the cone cut out by a system of inequalities. They also found a robust family of $\psi$ for $\mathfrak{sl}_2$ that satisfied positivity: $K^{\psi}_{\lambda,\mu}(t) \in \mathbb{N}[t]$. (They conjectured that the weights of Demazure modules would give positivity in the $\mathfrak{sl}_3$ case.)

There was a particularly large and vibrant group studying the LLT (ribbon) polynomials. Based on computer experiment conducted by Mark Shimozono, the LLT polynomial indexed by a $\lambda$ whose $k$-quotient consisted of rectangles agrees with the generalized Hall-Littlewood polynomial indexed by the same rectangles. (LLT= GHL) (Hall-Littlewood polynomials have Kostka polynomials as coefficients when expressed in the Schur basis.) While the left-hand-side makes sense in the non-rectangle realm, it does not for the right-hand-side (or at least cannot rely on the combinatorics of crystals that use the existence of the "combinatorial $R$" which we only have for rectangles). There were several proposed methods of showing these two families of polynomials agree that the group explored. One such was finding a bijection on tableaux that exchanged the statistic of energy with that of cospin.

This report was of particular interest to another group who had been discussing the $q, t$-symmetry of the Macdonald polynomials. Haiman-Haglund-Loehr have a combinatorial expression for Macdonald polynomials computed using statistics that extend m$aj$ and i$nv$, which energy and cospin mentioned above also extend. A statistic swapping bijection (extending the Foata bijection) would explain the $q, t$ symmetry. This problem had also been mentioned in the problem session on the first day. This group also worked on the conjectural Schur-positivity of the LLT polynomials.

Another open problem that the first LLT group discussed was putting a crystal structure on ribbon tableaux that is compatible with the spin statistic. In the rectangle case, this reduces to their main problem under focus. One member of the LLT=GHL group commented he probably would not have focused on that problem in the course of his own research, nor would he have had the chance to collaborate with these group members if not for the workshop. He also noted the combined strengths and viewpoints of the group complemented each other and this problem was the ideal meeting ground. They traded combinatorial viewpoints of ribbon tableaux and rigged configurations with statistics of spin and energy, searching for bijections between the two realms. They learned new ways of viewing the ribbon tableaux via both the frontier and abacus methods of going from a diagram to its core and quotient. The participants agreed they had a much better feel for the polynomials and the combinatorial objects used to compute them.

The graduate student who taught this group about the abacus method commented that she had been feeling unsure about continuing in math prior to the workshop. After being able to contribute something to her group (and meeting other mathematicians besides just her adviser, with whom she had always felt slow in comparison), she realized she was not slow and did have something to contribute to mathematics. She is now as energized as ever.

Many of the conjectures participants had ventured in their past research involved making a large number of computations on computer. One of our graduate student participants, Francois Descouens, is involved in the design of "MuPAD," a computer package for algebraic combinatorics. He helped other participants install this software on their computers and gave them tutorials in using it.

**Open Problems.** There were more problems posed than the groups had time to work on, such as extending various type $A$ constructions to other root systems. Another series of questions surrounded using lattice diagrams (not just Ferrers diagrams) to construct a family of symmetric polynomials that shares many remarkable properties with the Macdonald polynomials, but conspicuously lacks the property of being a basis, i.e. it is an overdetermined family. The operator $\nabla$, whose eigenfunctions are the Macdonald polynomials, was discussed, as well as the expansion of $\nabla e_n$.

In the open problem session, there was some discussion on the geometric interpretation of the objects under focus. However, this led more to clarification and discussion than to the posing of well-defined open problems.

A complete list of the open problems stated will be available on the workshop website shortly.

**Outlook.** We hope that significant progress will be made on the open problems and conjectures discussed at AIM, and will be continued in the collaborations begun at the workshop. We found many of the unorthodox sessions and ideas at AIM to be very useful for fostering discussion and collaboration.

While many participants mentioned that at other conferences, they skip talks to meet with collaborators, or occasionally ask authors questions about their papers, they were happy workshop time was set aside for expressly this purpose. Furthermore, because it included everyone, people were included in and got to participate in smaller discussions who would not have otherwise. The junior participants especially appreciated this.

In conclusion, the ARCC workshop on generalized Kostka polynomials was a definite success. Everyone participated in at least one small group session, and we expect many of these sessions to lead to longer-term collaborations and substantial progress in the field. We owe a large debt of gratitude to the AIM staff for organizing and facilitating the workshop and to AIM and the NSF for funding the conference.