The Isoperimetric Inequality for SL(n, Z)

organized by
Nathan Broaddus, Tim Riley, and Kevin Wortman

Workshop Summary

1. Progress and programs for solutions

During the week of September 8-12, 2008 we held a workshop at AIM to answer the following question.

Does $\mathrm{SL}(n,\mathbb{Z})$ *admit a quadratic isoperimetric inequality when* $n \geq 4$*?*

We made progress in addressing this problem from the three perspectives below.

1.1. Modifying horoballs. A standard approach to constructing a quasi-isometric model for $\mathrm{SL}(4,\mathbb{Z})$ is to start with the symmetric space $X = \mathrm{SL}(4,\mathbb{R})/\mathrm{SO}(4,\mathbb{R})$ and remove a union of open horoballs to get a space

$$Y = X - \bigcup_{r\in\mathcal{R}} H_r$$

where $\mathcal{R}$ is an $\mathrm{SL}(4,\mathbb{Z})$-equivariant collection of geodesic rays in X. Here each horoball H_r is constructed using a Buseman function corresponding to the ray $r \in \mathcal{R}$. Since X is nonpositively curved it satisfies a quadratic isoperimetric inequality. Therefore, given a loop in Y one may fill it with a disk $D \subset X$ whose area is quadratic in the length of the loop.

One would then like to push D out of each horoball H_r while fixing ∂D and increasing its area in a controlled (hopfully Lipschitz) manner. One difficulty with this approach is that one may find arbitrarily large sets of horoballs $\{H_1, \cdots, H_k\}$ with nonempty intersection

$$\bigcap_{i=1}^{k} H_i \neq \emptyset.$$

We considered an alternative collection $\mathcal{U}$ of subsets X with finite intersection properties, and such that

$$Y = X - \bigcup_{H\in\mathcal{U}} H$$

For the rings $A = \mathbb{Z}, \mathbb{R}$ Let N be the unipotent group

$$N(A) = \left\{ \begin{pmatrix} 1 & a_{11} & \cdots & a_{1n} \\ 0 & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} . \middle| a_{ij} \in A \right\}$$

The quotient $\mathrm{SL}(4,\mathbb{Z})\backslash X$ has a thick/thin decomposition where the noncompact thin part is isometric to a "semidirect product" $S \ltimes (N(\mathbb{Z})\backslash N(\mathbb{R}))$ where S is the Euclidean sector (and commutative monoid)

$$S = \left\{ \begin{pmatrix} e^{t_1} & 0 & 0 & 0 \\ 0 & e^{t_2} & 0 & 0 \\ 0 & 0 & e^{t_3} & 0 \\ 0 & 0 & 0 & e^{t_4} \end{pmatrix} \middle| t_i \in \mathbb{R}, \sum_{i=1}^{n} t_i = 0, t_1 \geq \cdots \geq t_4 \right\}.$$

Given $i, j \in \{1,2,3,4\}$, we let $S_{i=j}$ be all the points in S with $t_i = t_j$. For any subset $S' \subseteq S$, we let $F(S')$ be a fundamental domain in X for $S' \ltimes (N(\mathbb{Z})\backslash N(\mathbb{R}))$, and then we let $P(S') \leq \mathrm{SL}(4,\mathbb{Z})$ be the group that stabilizes $F(S')(\infty)$ and take $H(S') = P(S')F(S')$. Our set $\mathcal{U}$ of modified horoballs is the set of $\mathrm{SL}(4,\mathbb{Z})$ translates of $H(S)$, $H(S_{1=2})$, $H(S_{2=3})$, $H(S_{3=4})$, $H(S_{1=2=3})$, $H(S_{2=3=4})$, and $H(S_{1=2,3=4})$.

Now to prove that $\mathrm{SL}(4,\mathbb{Z})$ has a quadratic Dehn function, one has to show that the bondaries of each of the first four of modified horoballs listed above satisfies a quadratic Dehn function. With that knowledge, we can assume that our disc $D \subseteq X$ can be replaced by a disc with the same boundary, with roughly the same area, that lies either in the thick part of X or else in one of the last three types of modified horoballs listed above. Having a disc entirely in the thick part is our goal, so the next step is to analyze the geometry of the last three types of modified horoballs. These horoballs reflect the geometry of groups like $\mathrm{SL}(3,\mathbb{Z}) \ltimes \mathbb{Z}^3$.

1.2. Recursive combinatorial subdivision of spanning disks. A number of combinatorial ideas were explored. A suggestion of Kassabov is to consider a weighted infinte presentation of the Steinberg group $\mathrm{St}(4,\mathbb{Z})$ which is quasi-isometric to $\mathrm{SL}(4,\mathbb{Z})$. Then Steinberg group has an infite presentation with generators $e_{ij}(\lambda)$ where $1 \leq i \neq j \leq 4$ and $\lambda \in \mathbb{Z}$ relations meant to model the relations behavior of elementary matrices:

$$e_{ij}(\lambda)e_{ij}(\nu) = e_{ij}(\lambda+\nu) \tag{1}$$

$$[e_{ij}(\lambda), e_{kl}(\nu)] = 1 \tag{2}$$

$$[e_{ij}(\lambda), e_{jk}(\nu)] = e_{kl}(\lambda\nu) \tag{3}$$

Instead of assigning length 1 and area 1 to each generator and relation one should consider the length of $e_{ij}(\lambda)$ to be $K_1 \log(\lambda)$, the trianglar relations of type (1) should have area $K_2 \log(\lambda)\log(\nu)$, the quadrilateral relations of type (2) should have area $K_3 \log(\lambda)\log(\nu)$, and the pentagon relations of type (3) should have area $K_4 \log(\lambda)\log(\nu)$. One may then try to prove that with this weighted presentation $\mathrm{SL}(4,\mathbb{Z})$ has a quadratic isoperimetric inequality and that this weighted infinite presentation give the same isoperimetric inequality as a finite presentation up to linear constants.

An argument of this type shows great promise of answering the a related conjecture that $\mathrm{SL}(3,\mathbb{Z})$ has linear filling length.

1.3. Homotopy type of the asymptotic cone. A nonconstructive approach to the problem is to try to decribe the homotopy type of the asymptotic cone of $\mathrm{SL}(4,\mathbb{Z})$. Simple connectivity of every asymptotic cone of $\mathrm{SL}(4,\mathbb{Z})$ would imply a quadratic Dehn function [Riley2003].

The asymptotic cone $X = \mathrm{Cone}_\infty(\mathrm{SL}(4,\mathbb{R}))$ for $\mathrm{SL}(4,\mathbb{R})$ is a Euclidean $\mathbb{R}$-building that one gets by a construction of Bruhat and Tits from considering a (nondiscrete) valuation on the Robinson field which is a quotient of a subring of the hyperreal numbers [KSTT]. Since $\mathrm{SL}(4,\mathbb{Z})$ is quasi-isometrically embedded in $\mathrm{SL}(4,\mathbb{R})$ one may view $\mathrm{Cone}_\infty(\mathrm{SL}(4,\mathbb{Z}))$ to be a subset of X. In fact, one may describe $\mathrm{Cone}_\infty(\mathrm{SL}(4,\mathbb{Z}))$ as a complement of a set of horoballs $\mathcal{H}$ in X.

The Euclidean $\mathbb{R}$-building X us a union of subsets $A \in \mathcal{A}$ isometric to Euclidean space $\mathbb{R}^{4-1}$ called apartments. If one can identify the homotopy type of

$$A - \bigcap_{H \in \mathcal{H}} H$$

for every apartment $A \in \mathcal{A}$ and further understand the homotopy types of intersections of such sets then one might be able to show simple connectivity for $\mathrm{Cone}_\infty(\mathrm{SL}(4,\mathbb{Z}))$ giving a polynomial Dehn function.

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